Lecture 6: Exponential Correction to Saturation

Olalla A. Castro-Alvaredo

School of Mathematics, Computer Science and Engineering
Department of Mathematics
City, University of London

November 14, 2016

1. The result

- We will end these lectures on QFT methods and Measures of Entanglement by showing an explicit computation which shown the intricacies of the twist field approach and the analytic continuation in n.
- We will prove that the EE of a single interval of length $\ell > \xi$ has the following universal behaviour in massive QFT:

Universal Exponential Corrections to Saturation

$$S(\ell) - \lim_{\ell\to\infty} S(\ell) = -\frac{1}{8}\sum_{\alpha=1}^{N} K_0(2m_\alpha \ell) + O(e^{-3m_1\ell})$$

- That is, there are exponentially decaying corrections to saturation which are led by the mass of the lightest particle in the spectrum m_1.
- This results was shown first in Cardy, Castro-Alvaredo, Doyon (2008) and the proven by Doyon to hold for any 1+1 dimensional QFT (even non-integrable).

2. Starting Point

- Recall that

$$S(\ell) = -\lim_{n\to 1} \frac{\partial h(n)}{\partial n} \quad \text{with} \quad h(n) = \epsilon^{4\Delta_{\mathcal{T}}} \langle \mathcal{T}(0)\tilde{\mathcal{T}}(\ell)\rangle$$

- So the basic object we need to compute is the two-point function:

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(\ell)\rangle = \langle \mathcal{T}\rangle^2 + \sum_{\mu} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} (F_1^{\mathcal{T}|\mu}(\theta))^* (F_1^{\tilde{\mathcal{T}}|\mu}(\theta)) e^{-\ell m_\mu \cosh\theta}$$

$$+\frac{1}{2}\sum_{\mu_1\mu_2} \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\theta_2}{2\pi} (F_2^{\mathcal{T}|\mu_1\mu_2}(\theta_1,\theta_2))^* (F_2^{\tilde{\mathcal{T}}|\mu_1\mu_2}(\theta_1,\theta_2)) e^{-\ell m_{\mu_1}\cosh\theta_1 - \ell m_{\mu_2}\cosh\theta_2}$$

$$+\cdots$$

3. Some Simplifications

- We have just seen the most general expansion up to two-particle form factors.
- Let us consider now a simple case: a theory with a single particle in the spectrum.
- In that case we can label particles just by the copy number $j = 1 \ldots n$.
- We also know the twist field is a spinless field: one-particle form factors are rapidity-independent and they are all equal because all copies are identical: $F_1^{\mathcal{T}|\mu}(\theta) := F_1(n)$.
- Two-particle form factors only depend on rapidity differences: $F_2^{\mathcal{T}|\mu_1\mu_2}(\theta_1, \theta_2) := F_2^{ij}(\theta, n)$ and $F_2^{\tilde{\mathcal{T}}|\mu_1\mu_2}(\theta_1, \theta_2) := \tilde{F}_2^{ij}(\theta, n)$ with $\theta = \theta_1 - \theta_2$.
- Finally, recall that all form factors are zero at $n = 1$.

4. First Term: Saturation

- The first term in the expansion of the two-point function is the expectation value of twist fields. This is a function of n which is only known for free theories.
- This term characterizes saturation of EE for large sub-systems:

$$\lim_{\ell\to\infty} S(\ell) = -\lim_{n\to 1}\frac{\partial}{\partial n}\left(\epsilon^{4\Delta_{\mathcal{T}}}\langle\mathcal{T}\rangle^2\right) = -\frac{c}{3}\log\epsilon - 2\lim_{n\to 1}\frac{\partial\langle\mathcal{T}\rangle}{\partial n}$$

$$= -\frac{c}{3}\log(\epsilon m) - U \quad \text{with} \quad \langle\mathcal{T}\rangle = m^{2\Delta_{\mathcal{T}}} U_n$$

- and $U = 2\lim_{n\to 1}\frac{\partial U_n}{\partial n}$. Note that U is a universal constant in the sense that it does not depend on the cut-off ϵ, hence can be uniquely determined for each QFT.

5. Second Term: One-Particle Form Factor

- For a theory with a single particle the one-particle form factor contribution can be written simply as

$$n\,|F_1(n)|^2 \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} e^{-\ell m \cosh\theta} = \frac{n}{\pi}\,|F_1(n)|^2\, K_0(m\ell).$$

- This provides the leading correction to saturation of the two-point function, however it vanishes under differentiation w.r.t. n and limit $n \to 1$.
- This is because $F_1(1) = F_1(1)^* = 0$.
- This means that the one-particle form factors (if they are non-vanishing) will provide the leading correction to the Rényi entropies but no contribution to the EE.

6. Third Term: Two-Particle Form Factor

- For a theory with a single particle two-particle form factor sum can be simplified as:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(F_2^{ij}(\theta,n))^*(\tilde{F}_2^{ij}(\theta,n)) = n\sum_{j=1}^{n}(F_2^{1j}(\theta,n))^*(\tilde{F}_2^{1j}(\theta,n))$$

because all copies are identical. Using the identities we saw in the previous lecture:

$$n\sum_{j=1}^{n}(F_2^{1j}(\theta,n))^*(\tilde{F}_2^{1j}(\theta,n)) = n\left|F_2^{11}(\theta,n)\right|^2 + n\sum_{j=2}^{n}\left|F_2^{11}(-\theta+2\pi i(j-1),n)\right|^2$$

$$= n\left|F_2^{11}(\theta,n)\right|^2 + n\sum_{j=1}^{n-1}\left|F_2^{11}(-\theta+2\pi ij,n)\right|^2$$

- The derivative at $n=1$ of the term $\left|F_2^{11}(\theta,n)\right|^2$ will be zero because $F_2^{11}(\theta,1) = F_2^{11}(\theta,1)^* = 0$. So it will contribute to the Rényi entropies but not to the EE.

7. In Summary: Leading Correction to EE

- In summary, we need to compute

$$-\frac{1}{4}\lim_{n\to 1}\frac{\partial}{\partial n}\left(\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\int_{-\infty}^{\infty}\frac{d\beta}{2\pi}n\sum_{j=1}^{n-1}\left|F_2^{11}(-\theta+2\pi ij,n)\right|^2 e^{-2m\ell\cosh\frac{\theta}{2}\cosh\frac{\beta}{2}}\right)$$

  with $\theta=\theta_1-\theta_2$ and $\beta=\theta_1+\theta_2$.
- The integral in β can be carried out giving a Bessel function. So, we end up with:

$$-\lim_{n\to 1}\frac{\partial}{\partial n}\left(\int_{-\infty}^{\infty}\frac{d\theta}{(2\pi)^2}n\sum_{j=1}^{n-1}\left|F_2^{11}(-\theta+2\pi ij,n)\right|^2 K_0(2m\ell\cosh\frac{\theta}{2})\right)$$

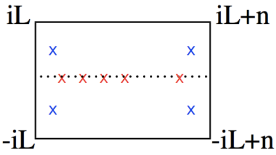
- In order to take the derivative, we need to somehow get rid of the sum up to $n-1$.
- A well-known way of doing this is to use the cotangent trick.

8.Cotangent Trick I

- The idea is that the sum may be replaced by a contour integral

$$\frac{1}{2\pi i}\oint dz \pi \cot(\pi z) s(z,\theta,n)$$

with $s(z,\theta,n) = \left|F_2^{11}(-\theta + 2\pi i z, n)\right|^2$, in such a way that the sum of the residues of poles of the cotangent enclosed by contour reproduces the original sum.



- Here the red crosses represent the poles of the cotangent at $z = 1, 2, \ldots, n-1$ and the blue crosses represent other poles in the contour due to the kinematic poles of the function $s(z,n)$ at $z = \frac{1}{2} \pm \frac{\theta}{2\pi i}$ and $z = n - \frac{1}{2} \pm \frac{\theta}{2\pi i}$.
- We shift $iL \to iL - \epsilon$ so as to avoid the pole at $z = n$. It includes $z = 0$ but this does not affect the result.

9.Cotangent Trick II

- Since $s(z, \theta, n)$ decays exponentially as $\mathrm{Im}(z) \to \pm\infty$ so we can show that the contributions to the contour integral of the horizontal segments vanish.
- The contribution of the vertical segments can be written as:

$$-\frac{1}{4\pi i}\int_{-\infty}^{\infty}(S(\theta-\beta)S(\theta+\beta)-1)\coth\frac{\beta}{2}s(\beta,\theta,n)d\beta$$

  where $\beta = 2\pi i z$ and $S(\theta)$ is the S-matrix. Here we used the property $s(z+n,\theta,n) = S(\theta-2\pi i z)S(\theta+2\pi i z)s(z,\theta,n)$.
- Note that this is zero for free theories. Its derivative at $n = 1$ is zero for similar reasons as before.
- Finally we are left with the contributions from the residues of the kinematic poles. They give:

$$\tanh\frac{\theta}{2}\mathrm{Im}\left(F_2^{11}(-2\theta+i\pi,n) - F_2^{11}(-2\theta+2\pi i n - i\pi,n)\right)$$

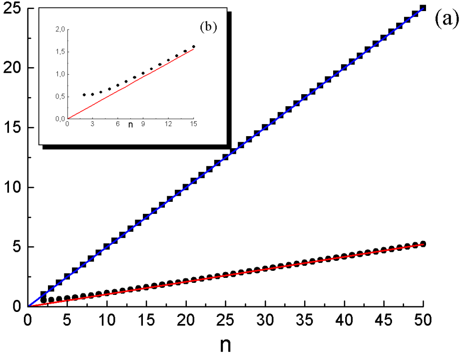
10. Derivative

- From these results, we already have an expression for the two-particle contribution to the Rényi entropies.
- However, our aim is to understand the derivative w.r.t. n of this function.
- We have already argued that the only two-particle contribution to the derivative comes from:

$$\mathrm{Im}\left(F_2^{11}(-2\theta+i\pi,n)-F_2^{11}(-2\theta+2\pi i n-i\pi,n)\right)\tanh\frac{\theta}{2}$$

- Based on previous observations, it would seem that this should be zero as $F^{11}(\theta,1)=0$. However, something special happens to this function as $n\to 1$ and $\theta\to 0$ simultaneously.
- This is due to the fact that as $n\to 1$ the two kinematic poles at $i\pi$ and $i\pi(2n-1)$ of the form factors collide giving a double pole for $\theta\neq 0$.
- For $\theta=0$ however, there are no poles and the function is simply $\frac{1}{2}$ (for all $n\neq 1$). It is however 0 at $n=1$!

11. A Picture: Better than 1000 Words



The sum $n\sum_{j=1}^{n-1}\left|F_2^{11}(-\theta+2\pi ij,n)\right|^2$ for $\theta=0$ in the Ising model (blue) and the sinh-Gordon model (red).

- Another way to write this is to note that near $n=1$ and $\theta=0$

$$\mathrm{Im}\left(F_2^{11}(-2\theta+i\pi,n)-F_2^{11}(-2\theta+2\pi i n-i\pi,n)\right)\tanh\frac{\theta}{2}$$

$$\sim -\frac{1}{2}\left(\frac{i\pi(n-1)}{2(\theta+i\pi(n-1))}-\frac{i\pi(n-1)}{2(\theta-i\pi(n-1))}\right)\sim\frac{\pi^2(n-1)}{2}\delta(\theta).$$

  near $n=1$ and $\theta=0$.

- Putting this result back into the θ integral and differentiating w.r.t. n we obtain the two-particle form factor contribution:

$$-\frac{1}{8}K_0(2m\ell)$$

- The result is striking for its simplicity. From the derivation we see that it follows from the pole structure of the FFs, which is universal.
- For this reason the same result can even be found for non-integrable models.