

Lecture 1A: Entanglement Measures in QFT

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8-19 February 2021

1. Entanglement in quantum mechanics

- A quantum system is in an entangled state if performing a localised measurement (in space and time) may instantaneously affect local measurements far away.
  A typical example: a pair of opposite-spin electrons:

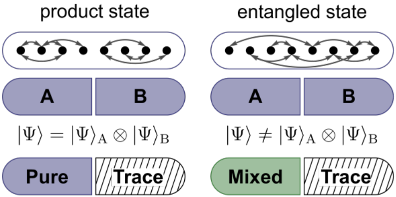
$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \;, \quad \langle \hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$$

- What is special: Bell's inequality says that this cannot be described by **local variables**.
- A situation that looks similar to $|\psi\rangle$ but without entanglement is a factorizable state:

$$|\hat{\psi}\rangle = \frac{1}{2}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right)$$

$$= \frac{1}{2}\left(|\uparrow\rangle + |\downarrow\rangle\right) \otimes \left(|\uparrow\rangle + |\downarrow\rangle\right)$$

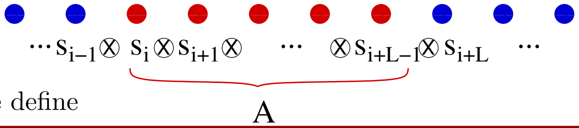
- These examples are extremely simple but what happens in extended many-body quantum systems?



- First of all, what provides a good measure of entanglement? [Plenio & Virmani'05]
  1. Entanglement monotone: no increase under LOCC
  2. Invariant under unitary transformations
  3. Zero for separable states
  4. (Usually) Non-zero for non-separable states
- Among others, the bipartite (or von Neumann) entanglement entropy, the Rényi entropies and the logarithmic negativity are all good measures of entanglement according to these properties.

2. Entanglement Entropy of Connected Regions

- Let us consider a spin chain of length N, subdivided into regions A and $\bar{A}$ of lengths L and $N-L$



then we define

Entanglement Entropy

$$S_A = -\mathrm{Tr}_{\mathcal{A}}(\rho_A \log \rho_A) \quad \text{with} \quad \rho_A = \mathrm{Tr}_{\bar{\mathcal{A}}}(|\Psi\rangle\langle\Psi|)$$

$|\Psi\rangle$ is a pure state of the system, ρ_A the reduced density matrix and $\mathcal{A}$ is the Hilbert space where A's degrees of freedom live.

- Other entropies may also be defined such as

Other Entropies

$$S_A^{\text{Rényi}} = \frac{\log(\mathrm{Tr}_{\mathcal{A}}\rho_A^n)}{1-n}, \quad S_A^{\text{Tsallis}} = \frac{1-\mathrm{Tr}_{\mathcal{A}}\rho_A^n}{n-1}$$

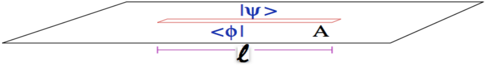
3. Computation in QFT: Replication

- The object $\mathrm{Tr}_{\mathcal{A}}\rho_A^n$ may be interpreted as a "replica" partition function:

$$\mathrm{Tr}_{\mathcal{A}}\rho_A^n = \mathrm{Tr}_{\mathcal{A}}({}_A\langle\phi|\rho_A|\psi_1\rangle_A\ {}_A\langle\psi_1|\rho_A|\psi_2\rangle_A \dots {}_A\langle\psi_{n-1}|\rho_A|\phi\rangle_A) = \frac{Z_n}{Z_1^n}$$

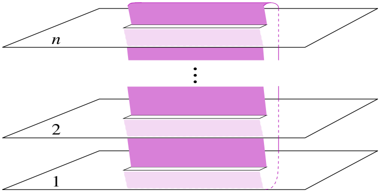
for n integer, in the scaling limit, Z_n is a partition function on an n-sheeted Riemann surface:

${}_A\langle\phi|\rho_A|\psi\rangle_A \sim$



$$\mathrm{Tr}_{\mathcal{A}}\rho_A^n \sim Z_n = \int [d\varphi]_{\mathcal{M}_n} \exp\left[-\int_{\mathcal{M}_n} d^2x\, \mathcal{L}[\varphi](x)\right]$$

$\mathcal{M}_n =$



4. Replica Trick

- We can express the bi-partite entanglement entropy directly in terms of this partition function as

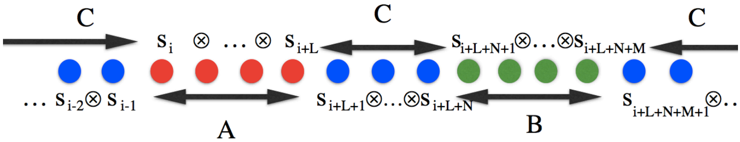
Replica Trick

$$S_A = -\mathrm{Tr}_{\mathcal{A}}(\rho_A \log \rho_A) = -\lim_{n\to 1} \frac{d}{dn} \mathrm{Tr}_{\mathcal{A}} \rho_A^n$$

- However, when computing this limit we need to extend our notion of "replica" to $n \geq 1$ and $n \in \mathbb{R}$.
- This is known as the analytic continuation problem and it is a difficult and not generally solved problem.
- Note that this is only a difficult problem when trying to obtain analytical results. If a numerical approach is available that allows for the diagonalization of ρ_A then any power (integer or not) of its eigenvalues can be computed.
- For this reason it is often easier to study the Rényi entropies with n integer than the bipartite EE.

5. Logarithmic Negativity (LN)

- The LN is a good measure of entanglement in pure and mixed states for non-complementary regions such as A and B [Vidal & Werner'01; Zyczkowski et al.'98; Plenio'05; Eisert'06]



Logarithmic Negativity

$$\mathcal{E} = \log \mathrm{Tr}_{\mathcal{A}\cup\mathcal{B}} |\rho_{A\cup B}^{T_B}| \quad \text{with} \quad \rho_{A\cup B} = \mathrm{Tr}_{\mathcal{C}}(|\Psi\rangle\langle\Psi|)$$

- It involves the trace norm: $\mathrm{Tr}_{\mathcal{A}\cup\mathcal{B}} |\rho_{A\cup B}^{T_B}| = \sum_i |\lambda_i|$ where λ_i are the eigenvalues of $\rho_{A\cup B}^{T_B}$.
- T_B represents partial transposition in sub-system B. Let e_i^A, e_i^B be bases in $\mathcal{A}$ and $\mathcal{B}$ then: $\langle e_i^A e_j^B | \rho_{A\cup B}^{T_B} | e_k^A e_l^B \rangle = \langle e_i^A e_l^B | \rho_{A\cup B} | e_k^A e_j^B \rangle$. The LN is basis-independent.

- There is also a replica approach to the LN [Calabrese, Cardy & Tonni'12]:

Replica Logarithmic Negativity

$$\mathcal{E}^n = \log \mathrm{Tr}_{\mathcal{A}\cup\mathcal{B}}(\rho_{A\cup B}^{T_B})^n \quad \text{then} \quad \mathcal{E} = \lim_{n\to 1} \mathcal{E}^{[}n_e]$$

where $\mathcal{E}^{n_e}$ means the function $\mathcal{E}^n$ for n even. This limit requires analytic continuation from n even to $n = 1$.

- There is also a partition function picture in this case. However, the n-sheeted Riemann surface is more complicated:

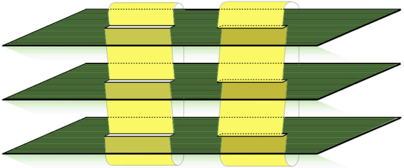


Fig. from Calabrese, Cardy & Tonni'12.

7. Measures of Entanglement in QFT: Why Bother?

- Entanglement growth is a key indicator of how effectively a quantum system can be simulated on a computer.
- Entanglement measures display remarkably universal features.
- A prime example are conformal field theories (CFT): [Holzhey, Larsen & Wilczek'94; Calabrese & Cardy'04; Calabrese, Cardy & Tonni'12].

EE of one Interval and LN of Adjacent Regions

$$S(\ell) = \frac{c}{3}\log\frac{\ell}{\varepsilon} \quad \text{and} \quad \mathcal{E}(\ell_1, \ell_2) = \frac{c}{4}\log\frac{\ell_1\ell_2}{\varepsilon(\ell_1+\ell_2)}$$

where ε is a non-universal short-distance cut-off. c is the central change. The EE and LN display both universal behaviour and dependence on universal features of the CFT.

- The dynamics of entanglement holds key information about the general dynamics of systems out of equilibrium.

8. Universality at and beyond Critical Points [GS]

- Short distance (CFT): Rényi Entropy for $0 \ll \ell \ll \xi$, logarithmic behaviour [Holzhey, Larsen & Wilczek'94; Calabrese & Cardy'04].

$$S_n(\ell) \sim |\partial A| \frac{(n+1)\,c}{12n} \log \frac{\ell}{\varepsilon}$$

where $|\partial A|$ is the number of boundary points.

- Large distance (massive QFT): $0 \ll \xi \ll \ell$, saturation

$$S_n(\ell) = -|\partial A| \frac{(n+1)\,c}{12n} \log(m_1 \varepsilon) + |\partial A|\, U_n + O\left(e^{-2m_1 \ell}\right)$$

Universal Exponential Corrections to Saturation

$$S(\ell) = -\frac{c}{3} \log(m_1 \varepsilon) + 2U_1 - \frac{1}{8} \sum_{\alpha=1}^{N} K_0(2\ell m_\alpha) + O\left(e^{-3m_1 \ell}\right)$$

m_α is the mass spectrum, $m_1 \propto \xi^{-1}$ is the smallest mass, N is the number of particles in the spectrum. [Cardy, OC-A & Doyon'08; Doyon'09].

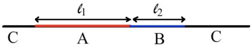
9. LN beyond Critical Points [GS]

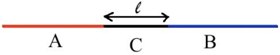
- Adjacent Regions (massive QFT): $0 \ll \xi \ll \ell$,

Universal Corrections to Saturation of the LN

$$\mathcal{E}^{\perp}(\ell) \sim -\frac{c}{4}\log(m_1\varepsilon) + \mathcal{E}_{\text{sat}} - \sum_{\alpha=1}^{N} \frac{2}{3\sqrt{3}\pi} K_0(\sqrt{3} m_\alpha \ell) \quad \ell_1 := \ell, \ell_2 \to \infty$$

where $m_1 \propto \xi^{-1}$ is the smallest mass scale in the theory, ϵ is a short distance cut-off and $\mathcal{E}_{\text{sat}}$ is a universal constant.





- Semi-infinite non-adjacent regions (massive QFT):

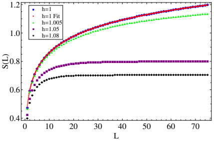
Universal Corrections to Saturation of the LN

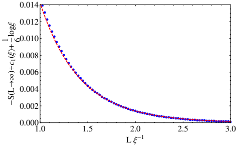
$$\mathcal{E}^{\dashv\vdash}(\ell) \sim \sum_{\alpha=1}^{N} \frac{(m_\alpha \ell)^2}{2\pi^2}\left[K_0(m_\alpha\ell)^2 + \frac{K_0(m_\alpha\ell)K_1(m_\alpha\ell)}{m_\alpha\ell} - K_1(m_\alpha\ell)^2\right]$$

[Blondeau-Fournier, O.C-A & Doyon'16]

$$H = -\frac{J}{2}\sum_{i=1}^{N}\left(\sigma_i^x\sigma_{i+1}^x + h\sigma_i^z\right)$$

- We may carry out the scaling limit of this theory in two different ways:
- Set $h = 1$ from the beginning: then $\xi = \infty$ and in the limit $N \to \infty$ this is a critical model.
- Take $h > 1$: $\xi \propto m^{-1}$ finite but large. Taking $N \to \infty$ while ℓ/ξ is finite we obtain Ising field theory.





- $S(\ell) = \frac{0.500003}{3}\log\ell + 0.478551$ for $h = 1$. For $h > 1$ saturation is reached.
- The corrections to saturation are exactly fitted by $\frac{1}{8}K_0(2m\ell)$.