Form Factor Programme: Exercises

1. Show that if the S-matrix of a diagonal theory has a representation of the form

$$S_{ab}(\theta) = \exp\left[\int_0^\infty \frac{dt}{t} g_{ab}(t) \sinh\frac{t\theta}{\pi}\right],$$

for some known function $g_{ab}(t)$, then the function

$$f_{ab}(\theta) = \exp\left[\int_0^\infty \frac{dt}{t}\frac{g_{ab}(t)}{\sinh t}\sin^2\left(\frac{it}{2}\left(1+\frac{i\theta}{\pi}\right)\right)\right]$$

is a minimal solution to the form factor equations, that is it satisfies:

$$f_{ab}(\theta) = S_{ab}(\theta) f_{ba}(-\theta) = f_{ab}(-\theta + 2\pi i).$$

You may assume that $f_{ab}(\theta) = f_{ba}(\theta)$ (parity invariance).

2. In the Ising model, the stress energy tensor (exceptionally) has only two non-vanishing form factors given by:

$$F_0^\Theta := \langle 0|\Theta|0\rangle = 2\pi m^2 \qquad F_2^\Theta(\theta_1, \theta_2) := -2\pi i m^2 \sinh\frac{\theta_1 - \theta_2}{2}.$$

Employing these form factors, write down the function $c(r)$ in as simplified a form as you can. From this formula show analytically that $\lim_{r\to 0} c(r) = \frac{1}{2}$. Plot your function $c(r)$ and compare it to the scaling function $c(R)$ of the TBA for the same theory. Check that although both have the same relevant features they are in fact different functions.

3. One way to identify the operator through its form factors is by looking at the short-distance behaviour of the two-point function. Numerically-speaking it is much easier to accurately identify the short-distance power-law behaviour of two-point functions by studying the form factor expansion of the logarithm of the two-point function, rather than the two-point function itself. In particular we expect that:

$$\log\left(\frac{\langle 0|\mathcal{O}(0)\mathcal{O}(r)|0\rangle}{\langle 0|\mathcal{O}|0\rangle^2}\right) \approx -4\Delta_{\mathcal{O}}\log r - 2\log\langle 0|\mathcal{O}|0\rangle + \cdots$$

where $\langle 0|\mathcal{O}|0\rangle$ is the one-point function or vacuum expectation value of the field $\mathcal{O}$. By definition, this is the zero-particle form factor and all higher particle form factors are proportional to it. Here we assume that $\mathcal{O}$ is self-conjugate so $\mathcal{O} = \mathcal{O}^\dagger$. Show that in a theory with a single particle spectrum and a diagonal S-matrix (for simplicity), the function above may be expanded as

$$\log\left(\frac{\langle 0|\mathcal{O}(0)\mathcal{O}(r)|0\rangle}{\langle 0|\mathcal{O}|0\rangle^2}\right) = \sum_{k=1}^\infty \int_{-\infty}^\infty \frac{d\theta_1\ldots d\theta_k}{k!(2\pi)^k} h_k(\theta_1,\ldots,\theta_k) e^{-mr\sum_{j=1}^k \cosh\theta_j}$$

where the functions h_k (usually called cumulants) can be expressed in terms of the usual form factors. For instance:

$$h_1(\theta) = |F_1^{\mathcal{O}}(\theta)|^2, \quad h_2(\theta_1, \theta_2) = |F_2^{\mathcal{O}}(\theta_1, \theta_2)|^2 - |F_1^{\mathcal{O}}(\theta_1)|^2 |F_1^{\mathcal{O}}(\theta_2)|^2$$

and so on. This is usually termed the cumulant expansion. By considering the small mr expansion of the formula above, show that, for a spinless field,

$$\Delta_{\mathcal{O}} = \frac{1}{2} \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \frac{d\theta_2 \ldots d\theta_k}{k!(2\pi)^k} h_k(0, \theta_2, \ldots, \theta_k).$$

Hint: For spinless fields the form factors and the funtions h_k only depend on rapidity differences. Use this fact to integrate out one rapidity in the cumulant expansion formula. After integration you should have a Bessel function which you can expand for small mr. You should then be able to get the formula for $\Delta_{\mathcal{O}}$ and with a bit more work you can even get a similar formula for $\log\langle 0|\mathcal{O}|0\rangle$.

4. The form factors of the operators μ and σ in the Ising model are given by:

$$F_{2k}^{\mu}(\theta_1, \ldots, \theta_{2k}) = i^k \langle 0|\mu|0\rangle \prod_{1\leq i<j\leq 2k} \tanh \frac{\theta_i - \theta_j}{2},$$

and

$$F_{2k+1}^{\sigma}(\theta_1, \ldots, \theta_{2k+1}) = i^k \langle 0|\sigma|0\rangle \prod_{1\leq i<j\leq 2k+1} \tanh \frac{\theta_i - \theta_j}{2}.$$

All other form factors are zero. Show that under clustering the following properties hold

$$\lim_{\lambda\to\infty} F_{2k}^{\mu}(\theta_1+\lambda, \ldots \theta_p+\lambda, \theta_{p+1}, \ldots \theta_{2k}) \propto \left\{ \begin{array}{ll} F_p^{\mu}(\theta_1, \ldots \theta_p) F_{2k-p}^{\mu}(\theta_{p+1}, \ldots \theta_{2k}) & \textbf{for } p \textbf{ even} \\ F_p^{\sigma}(\theta_1, \ldots \theta_p) F_{2k-p}^{\sigma}(\theta_{p+1}, \ldots \theta_{2k}) & \textbf{for } p \textbf{ odd} \end{array} \right.$$

and

$$\lim_{\lambda\to\infty} F_{2k+1}^{\sigma}(\theta_1+\lambda, \ldots \theta_p+\lambda, \theta_{p+1}, \ldots \theta_{2k+1}) \propto \left\{ \begin{array}{ll} F_p^{\sigma}(\theta_1, \ldots \theta_p) F_{2k+1-p}^{\mu}(\theta_{p+1}, \ldots \theta_{2k+1}) & \textbf{for } p \textbf{ odd} \\ F_p^{\mu}(\theta_1, \ldots \theta_p) F_{2k+1-p}^{\sigma}(\theta_{p+1}, \ldots \theta_{2k+1}) & \textbf{for } p \textbf{ even} \end{array} \right.$$

5. Employ the form factors of question 2 and question 4 and the Δ-sum rule to show that the field μ of question 4 can indeed be identified with one of the fields in the Ising model Kac table of conformal dimension 1/16.

6. We have seen in the lecture that it is common to compute FFs by starting with the ansatz:

$$F_k(\theta_1, \ldots, \theta_k) = H_k Q_k(x_1, \ldots, x_k) \prod_{i<j} \frac{F_{\mathbf{min}}(\theta_i - \theta_j)}{x_i + x_j},$$

where $x_i = e^{\theta_i}$ and we have written the ansatz for a theory with a single particle type and have used the fact that the minimal form factor is a function of rapidity differences.

Consider the sinh-Gordon model at the self-dual point. This is a theory with a single particle and scattering matrix

$$S(\theta) = \frac{\tanh\frac{1}{2}\left(\theta - \frac{i\pi}{2}\right)}{\tanh\frac{1}{2}\left(\theta + \frac{i\pi}{2}\right)}.$$

Using the fact that

$$F_{\mathbf{min}}(\theta + i\pi)F_{\mathbf{min}}(\theta) = \frac{\sinh\theta}{\sinh\theta + i},$$

and taking the factor of local commutatitivity $\omega = 1$, show, by plugging the ansatz above onto the kinematic residue equation, that the functions Q_k and the constants H_k satisfy

$$(-1)^k Q_{k+2}(-x, x, x_1, \ldots, x_k) = x D_k(x, x_1, \ldots, x_k) Q_k(x_1, \ldots, x_k), \qquad H_{k+2} = \frac{4H_k}{F_{\mathbf{min}}(i\pi)},$$

where $x_i = e^{\theta_i}$ and

$$D_k(x, x_1, \ldots, x_k) = \left(\sum_{j=0}^{k} (-1)^{j+1} \sin\frac{\pi j}{2} x^{k-j} \sigma_j^{(k)}\right)\left(\sum_{p=0}^{k} (-1)^{p} \cos\frac{\pi p}{2} x^{k-p} \sigma_p^{(k)}\right).$$

Recall that the elementary symmetric polynomials are given by the following generating function $\prod_{j=1}^{k}(x+x_j) = \sum_{j=0}^{k} x^{k-j}\sigma_j^{(k)}$ where $\sigma_j^{(k)}$ is the elementary symmetric polynomial of order j on k variables. For instance: $\sigma_1^{(3)} = x_1+x_2+x_3$, $\sigma_2^{(3)} = x_1x_2+x_2x_3+x_1x_3$ and $\sigma_3^{(3)} = x_1x_2x_3$. Hint: Write all functions involved in terms of the variables x_i.