Thermodynamic Bethe Ansatz: Exercises

1. In the lecture it was stated that the TBA equations may be obtained through a combination of the equations that relate the root density and overall density of states with the extremum condition for the free energy per unit length. Do the computation. Hint: You need to write down the equation $\frac{\delta f(\rho(\theta),\rho^{(r)}(\theta))}{\delta\rho^{(r)}(\beta)} = 0$. Start by employing the Bethe ansatz equation which relates ρ to $\rho^{(r)}$ to compute $\frac{\delta\rho(\theta)}{\delta\rho^{(r)}(\beta)}$. Note that these are functional derivatives so, for instance, $\frac{\delta\rho^{(r)}(\theta)}{\delta\rho^{(r)}(\beta)} = \delta(\theta-\beta)$.

2. Write down the exact (integral) expressions for the scaling function $c(R)$ of the free Fermion and free Boson theories. Show analytically how the $R \to 0$ limit indeed gives the values $c = \frac{1}{2}, 1$ as expected. Hint: Expand the L-function in powers of $e^{-\epsilon(\theta)}$.

3. The TBA equations can usually be solved easily by employing a simple numerical recipe. Write a code that solves the TBA equations for the Lee-Yang model. This is an integrable quantum field theory with a single particle spectrum and

$$S(\theta) = \frac{\tanh\frac{1}{2}\left(\theta + \frac{2\pi i}{3}\right)}{\tanh\frac{1}{2}\left(\theta - \frac{2\pi i}{3}\right)} \quad \text{and} \quad \varphi(\theta) = -\frac{4\sqrt{3}\cosh\theta}{1+2\cosh 2\theta}.$$

$S(0) = -1$ so we assume Fermionic statistics. Use your programme to plot the scaling function $c(R)$ and the L-functions $L(R)$ for various values of R. For small values you should find that the L-functions have a plateau at an irrational value which is exactly the solution of the constant TBA equation. Find this value analytically. The Lee-Yang model may be viewed as an integrable perturbation of the Lee-Yang minimal model of CFT. This is a non-unitary CFT with $c = -\frac{22}{5}$ and $\Delta = -\frac{1}{5}$ so that $c_{\mathbf{eff}} = \frac{2}{5}$. Hint: Explore values of R between 0.001 (UV) and 2 (IR). The numerical approach should be a recursive procedure where in the zero-th iteration $\epsilon^{(0)}(\theta) = mR\cosh\theta$ is the free solution and further solutions are obtained from the TBA equations as $\epsilon^{(k)}(\theta) = mR\cosh\theta - (\varphi * L^{(k-1)})(\theta)$ for $k = 0, 1, \ldots p$ for some maximum value p. This value is usually relatively large (e.g. $p \approx 50$) for small R but very small (e.g. $p \approx 10$) for large R. The correct solution is found when $\epsilon^{(k)}(\theta)$ does not change anymore under iteration.

4. Show that the constant TBA equations for the minimal A_n Toda theory with $N_{AB} = \delta_{AB} - 2(K^{-1})_{AB}$, where K is the Cartan matrix of A_n, have solutions

$$x_A = 1 + e^{-\epsilon_A} = \frac{\sin^2\left(\frac{\pi(A+1)}{n+3}\right)}{\sin\frac{\pi A}{n+3}\sin\frac{\pi(A+2)}{n+3}}.$$

Assume Fermionic statistics for all functions. Check that (at least for some fixed value of n) employing these solutions you obtain the right central charge from Roger's dilogarithm $c = \frac{2n}{(n+3)}$. Show that the constant TBA equations in terms of new variables $Q_A = \prod_{B=1}^{n} x_A^{K_{AB}}$ are

$$Q_{A+1}Q_{A-1} + 1 = Q_A^2.$$

Hint: Use the fact that, for A_n, $K_{AB} = 2\delta_{AB} - \delta_{A,B+1} - \delta_{A+1,B}$. Do not show that you get the correct central charge in general but just check a couple of cases numerically (say $n = 1$ and $n = 2$). The general proof involves using some special properties of Roger's dilogarithm.

5. Show how the $R \to 0$ limit of the TBA equations combined with the definition of the scaling function $c(R)$ gives rise to the expression of $c_{\mathbf{eff}}$ in terms of Roger's dilogarithm. Assume Fermionic statistics. Hint: It is hard to give hints for this question as the proof somehow only becomes obvious once you know which tricks to use. However, it is a very nice proof because it gives a very good illustration of the sort of tricks you can use to manipulate TBA equations. The key idea is to rewrite the $mR\cosh\theta$ term in both the TBA equation and the formula for $c(R)$ as $mR\cosh\theta = e^{\theta+x} + e^{-\theta+x}$ with $x = \log(mR/2)$ and then rewrite all formulae in terms of "shifted" variables by defining new functions $L_-(\theta) := L(\theta - x)$ and $\epsilon_-(\theta) = \epsilon(\theta - x)$. Terms of order e^{2x} can be neglected in the equations as we are interested in $x \to -\infty$. Assuming parity invariance $L(\theta) = L(-\theta)$, it is possible to express $c(R)$ as $\frac{6}{\pi^2}\int_x^\infty L_-(\theta)e^\theta$. The key is then to manipulate this formula by substituting e^θ by its expression from the "shifted" TBA equation and from the "shifted" derivative of the TBA equation. Zamolodchikov's original paper gives a hint of the derivation!