Lecture 2: Form Factor Programme

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- Like the TBA, the Form Factor Programme can be thought of as a means to carry out consistency checks of a given S-matrix (we will see how later).
- However, computing form factors gives us access to much more: the identification and classification of the local operator content and the computation of correlation functions of local fields of the IQFT.
- Form factors are matrix elements of a local field $\mathcal{O}(0)$ located at the origin between the ground state $\langle 0|$ and an arbitrary k-particle state $|k\rangle$: $F_k^{\mathcal{O}} = \langle 0|\mathcal{O}(0)|k\rangle$. They can be thought of a basic building blocks of correlation functions.
- Like for the S-matrix, integrability constraints the form factors very severely, to the extent that in integrable models they can often be computed exactly.

2. Definition

- Let $|\theta_1, \ldots, \theta_k\rangle_{\mu_1 \ldots \mu_k}$ a k-particle *in*-state. The particles have rapidities $\theta_1 > \cdots > \theta_k$ and quantum numbers $\mu_1 \ldots \mu_k$. Let $\mathcal{O}(0)$ be a local field located at the origin of space-time. Let $|0\rangle = (\langle 0|)^\dagger$ be the ground state (vacuum).

k-Particle Form Factor

$$F_k^{\mathcal{O}|\mu_1 \ldots \mu_k}(\theta_1, \ldots, \theta_k) := \langle 0|\mathcal{O}(0)|\theta_1, \ldots, \theta_k\rangle_{\mu_1 \ldots \mu_k}$$

- It is easy to "shift" operators away from the origin by using:

$$\langle 0|\mathcal{O}(\mathbf{x})|\theta_1, \ldots, \theta_k\rangle_{\mu_1 \ldots \mu_k} = \left(\prod_{j=1}^{k} e^{ip^\nu(\theta_j)x_\nu}\right) F_k^{\mathcal{O}|\mu_1 \ldots \mu_k}(\theta_1, \ldots, \theta_k).$$

- Note that $p^0(\theta_j) = m_{\mu_j} \cosh \theta_j$ and $p^1(\theta_j) = m_{\mu_j} \sinh \theta_j$.
- Under Hermitian conjugation:

$$_{\mu_1 \ldots \mu_k}\langle \theta_k \ldots \theta_1|\mathcal{O}(0)|0\rangle = (F_k^{\mathcal{O}^\dagger|\mu_1 \ldots \mu_k}(\theta_1, \ldots, \theta_k))^*$$

3. A Riemann-Hilbert Problem

- Form factors satisfy a set of equations which specify their monodromy properties (Watson's equations) and their pole structure (Residue equations).
- Altogether, these equations are enough to (almost) entirely fix the form factors (FFs).
- The first equation describes the effect of exchanging two particles. The second equation, also know as crossing relation specifies the properties of the FF under a $2\pi i$ rapidity shift.

Watson's equations

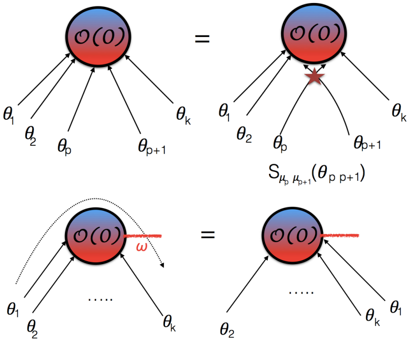
$$F_k^{\mathcal{O}|\ldots\mu_p\mu_{p+1}\ldots}(\ldots\theta_p,\theta_{p+1}\ldots) = S_{\mu_p\mu_{p+1}}(\theta_{p,p+1}) F_k^{\mathcal{O}|\ldots\mu_{p+1}\mu_p\ldots}(\ldots,\theta_{p+1},\theta_p,\ldots)$$

$$F_k^{\mathcal{O}|\mu_1\ldots\mu_k}(\theta_1+2\pi i,\ldots,\theta_k) = \omega F_k^{\mathcal{O}|\mu_2\ldots\mu_k\mu_1}(\theta_2,\ldots,\theta_k,\theta_1)$$

4. Watson's Equations in Pictures

$$F_k^{\mathcal{O}|\ldots\mu_p\mu_{p+1}\ldots}(\ldots\theta_p,\theta_{p+1}\ldots) = S_{\mu_p\mu_{p+1}}(\theta_{p,p+1})F_k^{\mathcal{O}|\ldots\mu_{p+1}\mu_p\ldots}(\ldots,\theta_{p+1},\theta_p,\ldots)$$

$$F_k^{\mathcal{O}|\mu_1\ldots\mu_k}(\theta_1+2\pi i,\ldots,\theta_k) = \omega F_k^{\mathcal{O}|\mu_2\ldots\mu_k\mu_1}(\theta_2,\ldots,\theta_k,\theta_1)$$



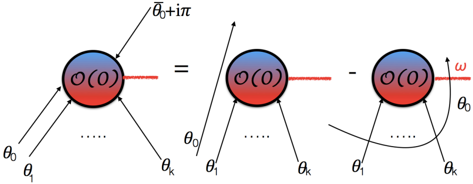
5. Kinematic Residue Equations

- Form factors posses kinematic poles when the rapidities of conjugate particles differ by $i\pi$.
- They provide a set of recursive equations relating $k+2$- to k-particle form factors which can be solved recursively.

Kinematic Residue Equations

$$\lim_{\bar{\theta}_0 \to \theta_0} (\bar{\theta}_0 - \theta_0) F_{k+2}^{\mathcal{O}|\bar{\mu}\mu\mu_1\ldots\mu_k}(\bar{\theta}_0 + i\pi, \theta_0, \theta_1, \ldots, \theta_k)$$

$$= i\left(1 - \omega \prod_{j=1}^{k} S_{\mu\mu_j}(\theta_{0j})\right) F_k^{\mathcal{O}|\mu_1\ldots\mu_k}(\theta_1, \ldots, \theta_k)$$



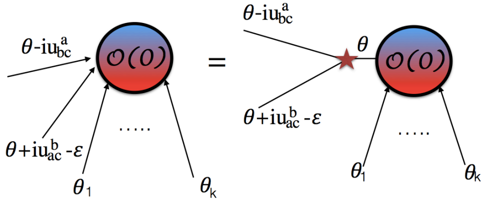
6. Bound State Residue Equations

- Form factors posses bound state poles if the S-matrix has.
- They provide a set of recursive equations relating $k+2$- to $k+1$-particle form factors which can be solved recursively.

Bound State Residue Equations

$$\lim_{\epsilon\to 0} \epsilon F_{k+2}^{\mathcal{O}|ab\mu_1\ldots\mu_k}(\theta + i\bar{u}_{ac}^{b} + \epsilon, \theta - i\bar{u}_{bc}^{a}, \theta_1, \ldots, \theta_k) = i\Gamma_{ab}^{c} F_{k+1}^{\mathcal{O}|c\mu_1\ldots\mu_k}(\theta, \theta_1, \ldots, \theta_k)$$

- Here $\bar{u}_{ac}^{b} = \pi - u_{ac}^{b}$ and iu_{ac}^{b} is a simple pole of the S-matrix $S_{ac}(\theta)$. Γ_{ab}^{c} is the square root of the residue of $S_{ab}(\theta)$ at iu_{ab}^{c} corresponding to the bound state c in the process $a+b\to c$.



7. Scaling and Asymptotics

- Relativistic Invariance: Under a Lorenz Boost rapidities experience a constant shift. Form factors scale as:

Relativistic Invariance

$$F_k^{\mathcal{O}|\mu_1\ldots\mu_k}(\theta_1+\lambda,\ldots,\theta_k+\lambda)=e^{s\lambda}F_k^{\mathcal{O}|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k)$$

where s is called the spin of the operator $\mathcal{O}$. As a consequence, the form factors of spinless operators are functions of rapidity differences only.

- Asymptotic Bounds: these are constraints to the asymptotic behaviour of the form factors which help with operator identification

Asymptotic Bounds

$$\lim_{\theta_i\to\infty} F_k^{\mathcal{O}|\mu_1\ldots\mu_i\ldots\mu_k}(\theta_1,\ldots,\theta_i,\ldots\theta_k)\propto e^{\alpha\theta_i} \quad with \quad \alpha\leq\Delta_{\mathcal{O}}$$

8. Clustering

- Cluster Decomposition Property: It has been observed and shown under special asumptions:

Momentum Space Cluster Property

$$\lim_{\theta_1,\ldots,\theta_p\to\infty} F_k^{\mathcal{O}_1|\mu_1\ldots\mu_p\mu_{p+1}\ldots\mu_k}(\theta_1,\ldots,\theta_p,\theta_{p+1}\ldots\theta_k)$$

$$\sim F_p^{\mathcal{O}_2|\mu_1\ldots\mu_p}(\theta_1,\ldots,\theta_p)F_{k-p}^{\mathcal{O}_3|\mu_{p+1}\ldots\mu_k}(\theta_{p+1},\ldots,\theta_k)$$

- The operators $\mathcal{O}_{1,2,3}$ may all be different if the theory has internal symmetries (e.g. the Ising model has $\mathbb{Z}_2$ symmetry). Clustering can then be used to systematically construct the FFs of new fields from the FFs of one original field.
- For many operators and theories, the three fields are the same. This can also be useful for instance to fix the value of the 1-particle form factor (a constant for spinless fields) from the asymptotics of the two-particle form factor.

- Locality: the locality of fields is an in-built assumption of the form factor equations. It may be expressed by saying that $[\mathcal{O}(\mathbf{x}), \mathcal{V}(\mathbf{y})] = 0$ if points $\mathbf{x}, \mathbf{y}$ are not causally connected.

(Semi) Locality

$$\mathcal{O}(\mathbf{y})\mathcal{V}(\mathbf{x}) = \begin{cases} \mathcal{V}(\mathbf{x})\mathcal{O}(\mathbf{y}) & \textbf{for} \quad \mathbf{x^1} > \mathbf{y^1} \\ \omega\mathcal{V}(\mathbf{x})\mathcal{O}(\mathbf{y}) & \textbf{for} \quad \mathbf{x^1} < \mathbf{y^1} \end{cases}$$

- The notion of locality can be generalised to semi-local fields by including an index of local commutatitivity ω.
- It can be generalized even further by considering more general symmetries:

Twist Fields

$$\mathcal{O}(\mathbf{y})\mathcal{V}(\mathbf{x}) = \begin{cases} \mathcal{V}(\mathbf{x})\mathcal{O}(\mathbf{y}) & \textbf{for} \quad \mathbf{x^1} > \mathbf{y^1} \\ \mathcal{V}'(\mathbf{x})\mathcal{O}(\mathbf{y}) & \textbf{for} \quad \mathbf{x^1} < \mathbf{y^1} \end{cases} \qquad \mathcal{V}' = \sigma\mathcal{V}\sigma^{-1}$$

- $\mathcal{O}$ is the twist field associated to the symmetry σ.

- A major problem within the FF programme is that there can be many distinct solutions to the same set of FF equations.
- Given a solution to the FF equations, how do we identify which local field $\mathcal{O}$ does it correspond to?
- How many distinct solutions to the FF equations are there?
- IQFTs are perturbations of CFT. It is expected that the operator content of IQFT is in one-to-one correspondence with the operator content of CFT.
- For some models (e.g. the Ising model) it has been possible to prove that the number of solutions to the FF equations is exactly the same as the number of CFT fields.
- Operators can be identified through the asymptotic properties of the form factors and through mainly three consistency checks: two-point functions short-distance asymptotics, Δ-sum rule and c-theorem.

- The main reason why FFs are such a powerful tool is that they provide the building blocks for the construction of every correlation function of the IQFT.
- The easiest correlators to compute are two-point functions such as $\langle 0|\mathcal{O}_1(0)\mathcal{O}_2(r)|0\rangle$.
- They can be expressed in terms of FFs by defining the following sum over a complete set of states:

$$P := \sum_{k=0}^{\infty} \sum_{\mu_1,\ldots,\mu_k}^{\ell} \int_{-\infty}^{\infty} \frac{d\theta_1 \ldots d\theta_k}{k!(2\pi)^k} |\theta_k \cdots \theta_1\rangle_{\mu_k \ldots \mu_1} \;{}_{\mu_1 \ldots \mu_k}\langle \theta_1 \cdots \theta_k|$$

ℓ is the number of particle species.

12. Correlation Functions and Form Factors

- Inserting the projector P between the two fields in a two-point function we can write:

Form Factor Expansion

$$\langle 0|\mathcal{O}_1(0)\mathcal{O}_2(r)|0\rangle = \sum_{k=0}^{\infty}\sum_{\mu_1,\ldots,\mu_k}^{\ell}\int_{-\infty}^{\infty}\frac{d\theta_1\ldots d\theta_k}{k!(2\pi)^k}F_k^{\mathcal{O}_1|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k)$$
$$\times F_k^{\mathcal{O}_2^\dagger|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k)^* e^{-r\sum_{j=1}^{k} m_j\cosh\theta_j}$$

- This is a rapidly convergent expansion for $m_j r \gg 1$. It is a large distance expansion.
- However, in many cases, it also provides a very good description of the short-distance behaviour, even if only few terms in the sum are included!
- This provides a way to test features of the underlying CFT by employing FFs of fields in the massive QFT.

13. Identifying the Conformal Dimension of Fields

- At short-distances we expect that

Short-Distance Decay of Correlators

$$\langle 0|\mathcal{O}_1(0)\mathcal{O}_1^\dagger(r)|0\rangle \approx r^{-4\Delta_\mathcal{O}}$$

where $\Delta_\mathcal{O}$ is the conformal dimension of $\mathcal{O}$. Sometimes, it is possible to use the FF expansion to test this with great precision.

- Another way of expressing this is to use the Δ-sum rule:

Sum Rule

$$\Delta_\mathcal{O} = -\frac{1}{2\langle\mathcal{O}\rangle}\int_0^\infty dr\, r\langle 0|\Theta(0)\mathcal{O}(r)|0\rangle_c$$

where Θ is the trace of the energy-momentum tensor (proportional to the perturbing field at criticality) and 'c' means the connected correlator.

14. c-Theorem

- A consequence of Zamolodchikov's c-theorem is that:

Central Charge

$$c = \frac{3}{2} \int_0^\infty dr\, r^3 \langle 0|\Theta(0)\Theta(r)|0\rangle_c$$

this can be used as a consistency check for the FFs of Θ.

- More generally, the c-theorem tells us that the function:

c-theorem

$$c(r) = \frac{3}{2} \int_r^\infty ds\, s^3 \langle 0|\Theta(0)\Theta(s)|0\rangle_c$$

has all properties of a c-function.

- It is monotonically decreasing from the UV (small r) to the IR (large r), its derivative vanishes at conformal critical points and it is positive-definite.

15. Solution Procedure

- For diagonal theories, there is a well-understood systematic approach to computing form factors of local fields.
- It starts with solving Watson's equations for a minimal two-particle form factor.
- Such a form factor satisfies:

$$F_{\mathbf{min}}^{ab}(\theta_1,\theta_2) = S_{ab}(\theta_1-\theta_2)F_{\mathbf{min}}^{ba}(\theta_2,\theta_1) = F_{\mathbf{min}}^{ab}(\theta_2+2\pi i,\theta_1)$$

  and solutions to these equations can be found systematically by employing the integral representation of the S-matrix.
- The minimal form factors can then be employed as building blocks for a more general ansatz

$$F_k^{\mathcal{O}|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k) = H_k^{\mathcal{O}|\mu_1\ldots\mu_k} Q_k^{\mathcal{O}|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k)\prod_{i<j}\frac{F_{\mathbf{min}}^{\mathcal{O}|\mu_i\mu_j}(\theta_i,\theta_j)}{(e^{\theta_i}+e^{\theta_j})^{\delta_{\mu_i,\bar{\mu}_j}}}$$

16. Form Factor Ansatz

- The ansatz

Form Factor Ansatz

$$F_k^{\mathcal{O}|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k) = H_k^{\mathcal{O}|\mu_1\ldots\mu_k} Q_k^{\mathcal{O}|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k) \prod_{i<j} \frac{F_{\mathbf{min}}^{\mathcal{O}|\mu_i\mu_j}(\theta_i,\theta_j)}{(e^{\theta_i}+e^{\theta_j})^{\delta_{\mu_i,\bar{\mu}_j}}}$$

works provided the functions Q_k satisfy certain properties.

- Watson's equations are automatically satisfied thanks to the properties of the minimal form factor and provided that the functions Q_k are symmetric in all rapidities and invariant under a $2\pi i$ shift of any rapidity.
- The functions Q_k are combinations of elementary symmetric polynomials on the variables $x_i = e^{\theta_i}$.
- The denominators $e^{\theta_i}+e^{\theta_j}$ capture the kinematic pole structure. Additional poles are present if there are bound states.