

Lecture 2A: Form Factor Programme for Twist Fields

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1. Preliminary Definitions

- In 1+1D QFT it is natural to write a k-particle in-state as

$$|\theta_1, \ldots, \theta_k|0\rangle_{\mu_1\ldots\mu_k} \qquad \text{with} \qquad \theta_1 > \cdots > \theta_k$$

  where $\{\theta_i\}$ are rapidities in terms of which the energy and momentum of each excitation are $e(\theta) = m\cosh\theta$ and $p(\theta) = m\sinh\theta$. $\mu_1\ldots\mu_k$ are quantum numbers. $|0\rangle = (\langle 0|)^\dagger$ the ground state.

- Let $\mathcal{O}(0)$ be a local field located at the origin of space-time:

k-Particle Form Factor of a Local Field

$$F_k^{\mathcal{O}|\mu_1\ldots\mu_k}(\theta_1, \ldots, \theta_k) := \langle 0|\mathcal{O}(0)|\theta_1, \ldots, \theta_k|0\rangle_{\mu_1\ldots\mu_k}$$

- We will also need the S-matrix. We will look only at the diagonal case:

Two Particle Scattering Matrix

$$|\theta_1, \ldots, \theta_i, \theta_{i+1}, \ldots \theta_k|0\rangle_{\mu_1\ldots\mu_i\mu_{i+1}\ldots\mu_k} :=$$

$$S_{\mu_i\mu_{i+1}}(\theta_i - \theta_{i+1})|\theta_1, \ldots, \theta_{i+1}, \theta_i, \ldots \theta_k|0\rangle_{\mu_1\ldots\mu_{i+1}\mu_i\ldots\mu_k}$$

2. Correlation Functions

- The main reason why form factors are a powerful tool is that they provide the building blocks for every correlation function in QFT. For example, two-point functions such as $\langle 0|\mathcal{O}_1(0)\mathcal{O}_2(r)|0\rangle$.
- They can be expressed in terms of FFs by defining the following sum over a complete set of states:

$$P := \sum_{k=0}^{\infty}\sum_{\mu_1,\ldots,\mu_k}^{\ell}\int_{-\infty}^{\infty}\frac{d\theta_1\ldots d\theta_k}{k!(2\pi)^k}|\theta_k\cdots\theta_1|0\rangle_{\mu_k\ldots\mu_1}\;{}_{\mu_1\ldots\mu_k}\langle 0|\theta_1\cdots\theta_k|$$

ℓ is the number of particle species.

- It is easy to "shift" operators away from the origin by using:

$$\langle 0|\mathcal{O}(\mathbf{x})|\theta_1,\ldots,\theta_k|0\rangle_{\mu_1\ldots\mu_k} = \left(\prod_{j=1}^{k} e^{ip^{\nu}(\theta_j)x_{\nu}}\right) F_k^{\mathcal{O}|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k).$$

- Under Hermitian conjugation:

$${}_{\mu_1\ldots\mu_k}\langle 0|\theta_k\ldots\theta_1|\mathcal{O}(0)|0\rangle = (F_k^{\mathcal{O}^{\dagger}|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k))^*$$

3. Correlation Functions from Form Factors

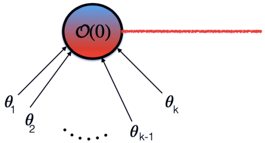
- Inserting the projector P between the two fields in a two-point function we can write:

Form Factor Expansion

$$\langle 0|\mathcal{O}_1(0)\mathcal{O}_2(r)|0\rangle = \sum_{k=0}^{\infty}\sum_{\mu_1,\ldots,\mu_k}^{\ell}\int_{-\infty}^{\infty}\frac{d\theta_1\ldots d\theta_k}{k!(2\pi)^k}F_k^{\mathcal{O}_1|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k)$$
$$\times F_k^{\mathcal{O}_2^\dagger|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k)^* e^{-r\sum_{j=1}^k m_j\cosh\theta_j}$$

- This is a rapidly convergent large-distance expansion $(m_j r \gg 1)$.
- But in many cases, it also provides a very good description of the short-distance behaviour, even with just few terms in the sum.
- This provides a way to test features of the underlying CFT by employing FFs of fields in the massive QFT.
- Since entanglement measures depend on correlators of BPTFs this kind of expansion becomes our main computational method in massive QFT.

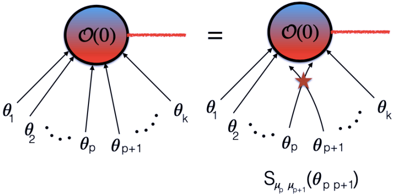
- In integrable QFT form factors satisfy a set of equations which specify their monodromy properties (Watson's equations) and their pole structure (Residue equations) [Karowski &Weisz'78; Smirnov'90s]. Good places to learn more [Smirnov's Book'92; Mussardo's Book'20]
- The programme was extended to BPTFs in [Cardy, OC-A & Doyon'08]
- In what follows I will call ω the semi-locality index as introduced in [Yurov & Zamolodchikov'91]. This represents the phase associated with exchanging the local field $\mathcal{O}$ with a particle-creating field. i.e. $\omega = -1$ for the field σ in the Ising model.



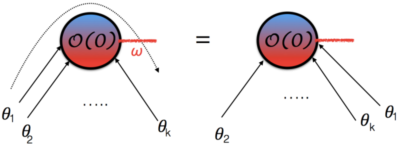
- The first Watson equation describes the effect of exchanging two particles. The second equation, also know as crossing relation specifies the properties of the FF under a $2\pi i$ rapidity shift.

4. Watson's Equations for Simple Twist-Fields

$$F_k^{\mathcal{O}|\ldots\mu_p\mu_{p+1}\ldots}(\ldots\theta_p,\theta_{p+1}\ldots) = S_{\mu_p\mu_{p+1}}(\theta_{p,p+1})F_k^{\mathcal{O}|\ldots\mu_{p+1}\mu_p\ldots}(\ldots,\theta_{p+1},\theta_p,\ldots)$$

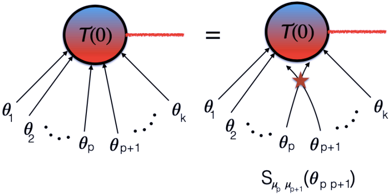


$$F_k^{\mathcal{O}|\mu_1\ldots\mu_k}(\theta_1+2\pi i,\ldots,\theta_k) = \omega F_k^{\mathcal{O}|\mu_2\ldots\mu_k\mu_1}(\theta_2,\ldots,\theta_k,\theta_1)$$

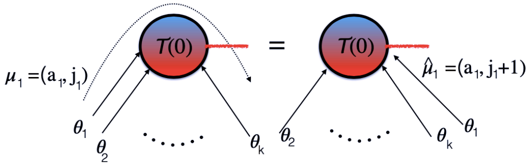


5. Watson's Equations for BPTFs

$$F_k^{\mathcal{T}|\ldots\mu_p\mu_{p+1}\ldots}(\ldots\theta_p,\theta_{p+1}\ldots) = S_{\mu_p\mu_{p+1}}(\theta_{p,p+1})F_k^{\mathcal{T}|\ldots\mu_{p+1}\mu_p\ldots}(\ldots,\theta_{p+1},\theta_p,\ldots)$$



$$F_k^{\mathcal{T}|\mu_1\ldots\mu_k}(\theta_1+2\pi i,\ldots,\theta_k) = F_k^{\mathcal{T}|\mu_2\ldots\mu_k\hat{\mu}_1}(\theta_2,\ldots,\theta_k,\theta_1)$$



- Now, the quantum numbers μ_i are double indices, labelling the particle and the copy.

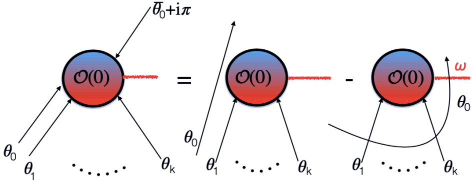
6. Kinematic Residue Equations for Simple Twist Fields

- Form factors posses kinematic poles when the rapidities of conjugate particles differ by $i\pi$.
- They provide a set of equations relating $k+2$- to k-particle form factors which can be solved recursively.

Kinematic Residue Equations

$$\lim_{\bar{\theta}_0 \to \theta_0} (\bar{\theta}_0 - \theta_0) F_{k+2}^{\mathcal{O}|\bar{\mu}\mu\mu_1\ldots\mu_k}(\bar{\theta}_0 + i\pi, \theta_0, \theta_1, \ldots, \theta_k)$$

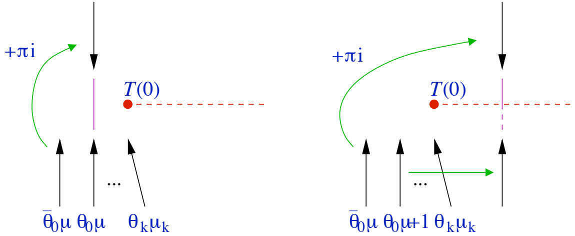
$$= i \left(1 - \omega \prod_{j=1}^{k} S_{\mu\mu_j}(\theta_{0j})\right) F_k^{\mathcal{O}|\mu_1\ldots\mu_k}(\theta_1, \ldots, \theta_k)$$



- For BPTFs, the kinematic residue equation separates in two:

$$\lim_{\bar{\theta}_0\to\theta_0}(\bar{\theta}_0-\theta_0)F_{k+2}^{\mathcal{T}|\bar{\mu}\mu\mu_1\ldots\mu_k}(\bar{\theta}_0+i\pi,\theta_0,\theta_1,\ldots,\theta_k)=iF_k^{\mathcal{T}|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k)$$

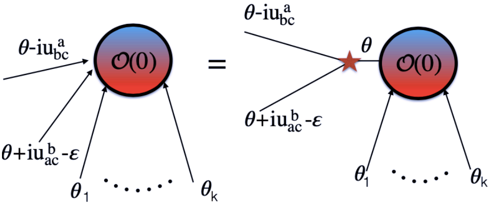
$$\lim_{\bar{\theta}_0\to\theta_0}(\bar{\theta}_0-\theta_0)F_{k+2}^{\mathcal{T}|\bar{\mu}\hat{\mu}\mu_1\ldots\mu_k}(\bar{\theta}_0+i\pi,\theta_0,\theta_1,\ldots,\theta_k)=-i\prod_{j=1}^{k}S_{\hat{\mu}\mu_j}(\theta_{0j})F_k^{\mathcal{T}|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k)$$



8. Bound State Residue Equations

- Form factors posses bound state poles if the S-matrix has.
- They provide a set of recursive equations relating $k+2$- to $k+1$-particle form factors which can be solved recursively.

$$\lim_{\epsilon\to 0} \epsilon F_{k+2}^{\mathcal{O}|ab\mu_1\ldots\mu_k}(\theta+i\bar{u}_{ac}^{b}+\epsilon, \theta-i\bar{u}_{bc}^{a}, \theta_1, \ldots, \theta_k) = i\Gamma_{ab}^{c} F_{k+1}^{\mathcal{O}|c\mu_1\ldots\mu_k}(\theta, \theta_1, \ldots, \theta_k)$$



- Here $\bar{u}_{ac}^{b} = \pi - u_{ac}^{b}$ and iu_{ac}^{b} is a simple pole of the S-matrix $S_{ac}(\theta)$. Γ_{ab}^{c} is the square root of the residue of $S_{ab}(\theta)$ at iu_{ab}^{c} corresponding to the bound state c in the process $a+b\to c$.
- The same equations hold for BPTFs if a, b live in the same copy.

9. Other Constraints: Scaling and Asymptotics

- Relativistic Invariance: Under a Lorenz boost rapidities experience a constant shift. Form factors scale as:

Relativistic Invariance

$$F_k^{\mathcal{O}|\mu_1\ldots\mu_k}(\theta_1+\lambda,\ldots,\theta_k+\lambda)=e^{s\lambda}F_k^{\mathcal{O}|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k)$$

where s is called the spin of the operator $\mathcal{O}$. Thus, the form factors of spinless operators (like $\mathcal{T}$) are functions of rapidity differences only.

- Asymptotic Bounds: these are constraints to the asymptotic behaviour of the form factors which help with operator identification [Delfino & Mussardo'95]

Asymptotic Bounds

$$\lim_{\theta_i\to\infty} F_k^{\mathcal{O}|\mu_1\ldots\mu_i\ldots\mu_k}(\theta_1,\ldots,\theta_i,\ldots\theta_k)\propto e^{\alpha\theta_i} \quad \text{with} \quad \alpha\leq\Delta_{\mathcal{O}}$$

10. Momentum Space Clustering

- Momentum Space Cluster Decomposition Property: It has been observed and shown under special assumptions [Delfino, Cardy & Simonetti'96]:

Momentum Space Cluster Property

$$\lim_{\theta_1,\ldots,\theta_p\to\infty} F_k^{\mathcal{O}_1|\mu_1\ldots\mu_p\mu_{p+1}\ldots\mu_k}(\theta_1,\ldots,\theta_p,\theta_{p+1}\ldots\theta_k)$$

$$\sim F_p^{\mathcal{O}_2|\mu_1\ldots\mu_p}(\theta_1,\ldots,\theta_p)F_{k-p}^{\mathcal{O}_3|\mu_{p+1}\ldots\mu_k}(\theta_{p+1},\ldots,\theta_k)$$

- The operators $\mathcal{O}_{1,2,3}$ may all be different if the theory has internal symmetries (i.e. the Ising model has $\mathbb{Z}_2$ symmetry). Clustering can then be used to systematically construct the FFs of new fields from the FFs of one original field.

- For many operators and theories, the three fields are the same. This can also be useful for instance to fix the value of the 1-particle form factor (a constant for spinless fields) from the asymptotics of the two-particle form factor.

11. Generalizations

- An interesting observation is that if we combine the equations for simple twist fields with those for BPTFs we can find a new set of form factor equations which describe composite fields $:\varphi\mathcal{T}:$.
- For instance, in the Ising model we could define the field $:\sigma\mathcal{T}:$ which combines $\mathbb{Z}_2$ and $\mathbb{Z}_n$ symmetries.
- These fields have found a recent application in the context of entanglement. They can be used to compute the symmetry resolved entanglement entropy and a form factor programme has been formulated for them [Horváth & Calabrese'20]