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PURELY ELASTIC SCATTERING THEORIES AND THEIR ULTRAVIOLET LIMITS

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We use the thermodynamic Bethe ansatz to find the finite-size corrections to the ground-state energy in an arbitrary (1 + 1)-dimensional purely elastic scattering theory. The leading finite-size effects are characterized by $\tilde{c} = c - 12d_0$, where c and d_0 are the central charge and the lowest scaling dimension, respectively, of the (possibly nonunitary) CFT describing the ultraviolet limit of the massive scattering theory. After presenting the purely elastic S-matrix theories that emerged in recent discussions of perturbed CFTs, we calculate their finite-size scaling coefficient $\tilde{c}$. Our results show that the UV limits of the "minimal" S-matrix theories are the unperturbed CFTs in question. On the other hand, the S-matrices which have been suggested to describe affine Toda field theories, differing from the minimal S-matrices by coupling-dependent factors, are seen to have free bosonic CFTs as their UV limits. We also discuss some interesting properties of $\tilde{c}$. In particular, we suggest that $\tilde{c}$ is a measure of the number of degrees of freedom of an arbitrary two-dimensional CFT.

1. Introduction

Recently there has been renewed interest in (1 + 1)-dimensional S-matrix theory. Starting with some remarkable papers by A.B. Zamolodchikov [1, 2], various groups [3–7] have used a counting argument by Zamolodchikov and explicit calculations in a few special cases [8, 9] to show that certain relevant perturbations of two-dimensional conformal field theories (CFTs) have nontrivial integrals of motion (IMs). In all cases considered so far there is a single perturbing parameter λ, and for at least one sign of λ the perturbed theory contains only massive excitations (with the mass scale of the theory being some function of λ which goes to zero as $\lambda \to 0$). The massive theory can then be described by a factorizable

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S-matrix [10]. The task of constructing the "minimal" (see sect. 2) part of the S-matrix of a factorizable theory simplifies considerably if all reflection amplitudes vanish. Using the bootstrap principle (sect. 2) and a few additional assumptions [2, 6], it is possible to construct a minimal reflectionless S-matrix with the same spectrum of IMs as that of the perturbed CFT, and this construction seems to be unique. Therefore it seems very likely that the minimal S-matrix thus constructed is indeed the minimal part of the S-matrix of the perturbed CFT. Further evidence is provided by recent numerical simulations; calculations of the excitation spectrum of certain lattice systems, which should renormalize to some of the perturbed CFTs in question, confirm at least the lowest mass ratios predicted by the S-matrix theories [11].

The (Lorentz) spins of the IMs in the perturbed CFTs considered in the literature follow a remarkable pattern: As far as they can be determined by present methods, they are the exponents of some affine Lie algebra, modulo its Coxeter number [12]. This affine Lie algebra, belonging to the list $A_n^{(1)}, A_{2n}^{(2)}, D_n^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$, is the same one on which a free-field representation [13, 14] (alternatively coset construction [15]) of the corresponding *unperturbed* CFT is based. Therefore the relation to affine Lie algebras is not totally unexpected. (The family of S-matrix theories related to the algebras $A_{2n}^{(2)}$, which are the only twisted, nonsimply laced one in the above list, is somewhat special, as will be explained in sect. 3.) One finds the connection to affine algebras even more striking when considering the structure of the S-matrix theories that have been proposed [3–7, 16] to describe perturbed CFTs. However, at present there is no understanding of the manifestation of the affine algebra structure in the details of these S-matrix theories, details on which we will elaborate in sect. 3.

There are other massive quantum field theories (QFTs) which are – in this case *by definition* – related to affine Lie algebras. These are the affine Toda field theories (ATFTs). The lagrangian of the Toda theory based on an affine Lie algebra of rank r, describes r massive scalar fields interacting through an exponential potential specified by the root system of the affine algebra [17, 18]. The simplest example is the "sinh-Gordon" model based on $A_1^{(1)}$. The ATFTs are known to be integrable at the classical level, again with the spins of the IMs given by the exponents of the affine Lie algebra, and there is increasing evidence [19] that they are integrable at the quantum level as well, at least those ATFTs based on simply laced affine algebras. In the latter cases low-order perturbation theory indicates [5, 16, 20] that the classical mass ratios, obtained by expanding the classical potential around its minimum, do not get renormalized in the quantum theory. These mass ratios exactly coincide with those in the minimal S-matrices that are candidates for describing perturbed CFTs. Naturally, therefore, the same minimal S-matrices have been suggested for the Toda theories [5, 16, 21]. However, the Toda lagrangians depend on a coupling constant, and so their S-matrices should also depend on it. In particular, when the coupling constant tends to zero, the S-matrices should become trivial, corresponding to theories of free massive bosons. This observation has led the authors of refs. [5, 16, 21] to conjecture that the S-matrices of the quantum ATFTs are the minimal S-matrices multiplied by certain coupling-dependent factors, the so-called [5] Z-factors. These factors do not have poles in the "physical strip" (see sect. 2), and so they do not introduce new particles into the theory. Nevertheless, their presence does of course change the dynamics of the theory.

The interesting question arises, whether a given massive perturbed CFT exhibiting IMs with spins that are the exponents of a certain affine Lie algebra, is described by the minimal S-matrix related to that algebra, or by the ATFT S-matrix (Z-factors included) at a specific coupling. (According to the conjectures in the literature [5, 16], the Z-factors are nontrivial for all values of the (real) coupling.) There is of course also a third possibility, that the perturbed CFT is described by none of the above two S-matrix theories: Either the minimal S-matrix is already wrong, because some basic assumptions leading to the minimal solution – e.g. that of reflectionless scattering – are not applicable, or the minimal S-matrix is correct but should be multiplied by different Z-factors (still consistent with the general constraints of S-matrix theory and the requirement that they do not introduce new masses into the theory). Based on the free-field representation [13, 14] of the unperturbed CFT, several authors [8, 9, 22] argued in favor of a relation between perturbed CFTs and ATFTs with *imaginary* coupling. At first sight, this proposal looks problematic, because generically a Toda theory with an imaginary coupling does not have a real lagrangian (the single exception being the ATFT of $A_1^{(1)}$, where replacing a real by an imaginary coupling corresponds to going from the sinh-Gordon to the sine-Gordon model). Even ignoring this problem, one would expect the ATFTs with an imaginary coupling to have solitons in their quantum spectrum – as in the sine-Gordon model – seemingly destroying the coincidence of the mass spectra of the perturbed CFT and the ATFT related to the same algebra. Nevertheless, perturbed CFTs might well be related to some *modified* imaginary-coupling affine Toda field theories, in which the above two problems are solved simultaneously. In fact, certain truncations of the sine-Gordon model at special values of the coupling have been proposed [23–25] to describe the $\phi_{(1,3)}$ perturbation of the Virasoro minimal models*. Although the sine-Gordon model is special in that the first mentioned problem does not occur, one might hope that also in general a suitable truncation of the Hilbert space of the imaginary-coupling ATFT would restore its unitarity, in analogy to the Feigin–Fuchs construction of CFTs [13].

In an attempt to answer the question raised in the previous paragraph, we analyze in the present paper the short-distance (UV) behavior of the purely elastic

* For details on the terminology and results of CFT used throughout this paper we refer the reader to the recent reviews of the subject by Cardy and Ginsparg [26] and references therein.

S-matrix theories (both minimal and nonminimal) that were recently discussed in connection with perturbed CFTs. We use the thermodynamic Bethe ansatz technique, following the treatment of Zamolodchikov [27] of the perturbed three-state Potts and Yang–Lee CFTs. In this approach, given a purely elastic S-matrix, one can explicitly evaluate the finite-size scaling coefficient $\tilde{c}$ (see sect. 4), characterizing the CFT at the UV limit of the given S-matrix theory. Our results for $\tilde{c}$ show that the UV limits of the minimal S-matrix theories are the CFTs in question. The UV limits of the S-matrices with the Z-factors proposed in the literature are seen to be theories of free massless bosons, supporting the conjecture that these S-matrices describe real-coupling affine Toda field theories.

The paper is organized as follows. In sect. 2 we review the general theory of purely elastic scattering in $1+1$ dimensions, and describe how one can construct purely elastic scattering theories using the bootstrap principle. In sect. 2 we discuss particular scattering theories in detail. This section contains some new results on $D_n^{(1)}$-related S-matrix theories, but it is mainly intended to summarize the data of the already known diagonal S-matrices, which we will need later. An outline of the thermodynamic Bethe ansatz method, leading to the useful result eq. (67) that allows a direct calculation of $\tilde{c}$ in terms of the S-matrix data, is presented in sect. 4. In sect. 5 we use the results of sect. 4 to calculate the finite-size scaling coefficients of the S-matrix theories described in sect. 3. In sect. 6 we discuss our results, and present a conjecture to which they have led us, namely, that $\tilde{c} = c - 12d_0$ is a measure of the number of degrees of freedom of an arbitrary two-dimensional (modular invariant) rational CFT.

2. Review of purely elastic scattering theory

The integrable deformations of CFTs studied in refs. [1–7, 16, 20] have led to *diagonal* factorizable S-matrix theories. Such scattering theories are also known as *purely elastic*. In the following we will summarize some basic facts about these theories, following the conventions of Zamolodchikov [2]. Nondiagonal theories will be briefly mentioned in sect. 6.

As remarked earlier, factorizability is a consequence of the existence of nontrivial IMs [10]. In a parity invariant $(1+1)$-dimensional QFT the existence of just one local IM* of Lorentz spin greater than one, in addition to energy–momentum, implies [28] that the *set of momenta* – hence, in particular, the number of particles – is conserved in all scattering processes (more on IMs later in this section). All that can happen is a reshuffling of momenta between particles of the same mass, possibly a change of internal quantum numbers, and a time delay (or advance) compared to the free case. The requirement that the unitary transforma-

* Although in all theories with one nontrivial IM there are known or at least conjectured to be infinitely many IMs.

tions generated by the nontrivial IMs (these transformations act like momentum-dependent translations) commute with the S-matrix, then implies that an arbitrary scattering amplitude can be decomposed into two-particle amplitudes. Consistency of different ways of decomposing an amplitude into two-particle amplitudes leads to cubic relations for the two-particle amplitudes $\mathscr{S}_{ab}^{cd}$ (here a, b denote the incoming and c, d the outgoing particles). These cubic relations are essentially the famous "Star–Triangle" or "Yang–Baxter equations" [10].

For a *diagonal* S-matrix not only the set of momenta, but the momentum and all the quantum numbers of each individual particle are conserved in every scattering process. The conservation of the individual momenta implies that the identity- and the $\mathscr{T}$-matrix in $\mathscr{S} = \mathbb{1} + i\mathscr{T}$ give the same momentum-conserving δ-functions when sandwiched between states. The usual analyticity requirements of QFT can therefore be expressed directly in terms of $\mathscr{S}$, after factoring out the momentum δ-functions. (The S-matrix $\mathscr{S}$ will always be understood to have the δ-functions factored out already.) The scattering of particles a and b is then described by the S-matrix element $S_{ab} \equiv \mathscr{S}_{ab}^{ab}$, which for physical momenta is just the exponential $e^{i\delta_{ab}}$ of the (momentum-dependent) phase shift δ_{ab} of the in-state with respect to the out-state. Explicitly, with an obvious notation for states, the definition of S_{ab} reads

$$|a(\theta_a)b(\theta_b)\rangle_{\text{in}} = S_{ab}(\theta_a, \theta_b)|a(\theta_a)b(\theta_b)\rangle_{\text{out}} . \tag{1}$$

Here we have introduced the rapidity θ, which provides a convenient way of parametrizing the momentum of a particle in 1 + 1 dimensions:

$$(p^0, p^1) = (m\cosh\theta, m\sinh\theta) . \tag{2}$$

Note that the Yang–Baxter equation is trivially satisfied for a diagonal S-matrix.

In terms of the Mandelstam variable $s = (p_a + p_b)^2$, $S_{ab}(s)$ has cuts, required by two-particle unitarity, along $s \leqslant (m_a - m_b)^2$ and $s \geqslant (m_a + m_b)^2$ in the complex s-plane. Since there is no particle production and any scattering amplitude factorizes into two-particle amplitudes, multi-particle unitarity is not expected to imply the existence of further cuts. "Anomalous thresholds" [29] in (1 + 1)-dimensional theories give rise to higher order poles – not to cuts as in higher dimensions – which will be discussed later. We therefore assume that the above cuts are the only cuts of $S_{ab}(s)$. They are purely kinematic and can be eliminated by considering S_{ab} to be a function of the relative rapidity $\theta_{ab} = |\theta_a - \theta_b|$ of the scattering particles, using

$$s = s(\theta_{ab}) = m_a^2 + m_b^2 + 2m_a m_b \cosh\theta_{ab} . \tag{3}$$

Physical values of s – the upper side of the cut along $s \geqslant (m_a + m_b)^2$ – are mapped to positive values of θ_{ab}, and the bound state region $(m_a - m_b)^2 < s <$

$(m_a + m_b)^2$ is mapped onto the segment $\mathrm{Re}\,\theta_{ab} = 0$ of the *physical strip* $0 < \mathrm{Im}\,\theta_{ab} < \pi$. Poles in the s-plane give rise to poles in the θ-plane; so we will assume that $S_{ab}(\theta)$ is a meromorphic function of θ. The assumption of real analyticity in the s-plane, $\mathscr{S}(s) = \mathscr{S}^*(s^*)$, translates into $\mathscr{S}(\theta) = \mathscr{S}^*(-\theta^*)$; in particular $S_{ab}(\theta)$ is real on the imaginary θ-axis. Using real analyticity the unitarity of the S-matrix, $\mathscr{S}(\theta)\mathscr{S}^\dagger(\theta) = \mathbb{1}$ for physical (real positive) values of θ, becomes $\mathscr{S}(\theta)\mathscr{S}^{\mathrm{T}}(-\theta) = \mathbb{1}$. By analytic continuation this latter equation then holds for all values of θ. In the purely elastic case, unitarity therefore reads

$$S_{ab}(\theta) S_{ab}(-\theta) = 1 \tag{4}$$

for all particles a, b in the model. We assume our theory to be parity invariant. Then $S_{ab}(\theta) = S_{ba}(\theta)$. By charge-conjugation symmetry we have

$$S_{\bar{a}\bar{b}}(\theta) = S_{ab}(\theta)\,. \tag{5}$$

Crossing symmetry, $S_{a\bar{b}}(s) = S_{\bar{a}b}(s) = S_{ab}(2m_a^2 + 2m_b^2 - s)$, becomes simply

$$S_{a\bar{b}}(\theta) = S_{\bar{a}b}(\theta) = S_{ab}(i\pi - \theta)\,, \tag{6}$$

where $\bar{b}$ denotes the antiparticle of b. Note that eqs. (4) and (6) imply that S_{ab} is a $2\pi i$-periodic function of θ.

In a local QFT we expect scattering amplitudes to be polynomially bounded in the momenta (this follows [30] from the Wightman axioms). It is a nice exercise to show (see ref. [31] for a similar problem) that under this assumption any meromorphic, real analytic (in the sense used above), $2\pi i$-periodic function $f(\theta)$ satisfying the unitarity condition (4) must be of the form

$$f(\theta) = \prod_{\alpha \in \mathrm{A}} f_\alpha(\theta)\,, \tag{7a}$$

where

$$f_\alpha(\theta) = \frac{\sinh \frac{1}{2}(\theta + i\alpha\pi)}{\sinh \frac{1}{2}(\theta - i\alpha\pi)}\,, \tag{7b}$$

for A a set of complex numbers invariant under complex conjugation*. We will assume that all poles occur on the imaginary θ-axis, i.e. there are no unstable particles. Then every α is real and we can choose $-1 < \alpha \leqslant 1$. The $f_\alpha(\theta)$ with α in this range are therefore the basic building blocks of our purely elastic scattering theories. Note that $f_\alpha(\theta)$ has a simple pole of residue $2i \sin \alpha\pi$ at $\theta = i\alpha\pi$, and a

* The functions excluded by polynomial boundedness are products of functions of the form $\exp[ia \sinh(2n+1)\theta]$, where a is real and n an integer.

simple zero at $\theta = -i\alpha\pi$. Further useful properties are

$$
\begin{aligned}
f_\alpha(\theta) &= f_{\alpha+2}(\theta) = f_{-\alpha}(-\theta)\,,\\
f_\alpha(\theta) f_{-\alpha}(\theta) &= 1\,,\\
f_\alpha(i\pi - \theta) &= -f_{1-\alpha}(\theta)\,,\\
f_\alpha(\theta - i\pi\beta) f_\alpha(\theta + i\pi\beta) &= f_{\alpha-\beta}(\theta) f_{\alpha+\beta}(\theta)\,,\\
f_0(\theta) \equiv -f_1(\theta) &\equiv 1\,.
\end{aligned}
\tag{8}
$$

If at least one of the particles a or b is its own antiparticle, the additional constraint of crossing invariance,

$$
S_{ab}(\theta) = S_{ab}(i\pi - \theta)\,,
\tag{9}
$$

implies (cf. ref. [4]) that up to a sign $S_{ab}(\theta)$ must be a product of functions of the form

$$
F_\alpha(\theta) = f_\alpha(\theta) f_\alpha(i\pi - \theta) = \frac{\sinh\theta + i\sin\alpha\pi}{\sinh\theta - i\sin\alpha\pi} = \frac{\tanh\frac{1}{2}(\theta + i\alpha\pi)}{\tanh\frac{1}{2}(\theta - i\alpha\pi)}\,.
\tag{10}
$$

These functions satisfy

$$
\begin{aligned}
F_\alpha(\theta) &= F_{\alpha+2}(\theta) = F_{1-\alpha}(\theta) = F_{-\alpha}(-\theta)\,,\\
F_\alpha(\theta) F_{-\alpha}(\theta) &= 1\,,\\
F_\alpha(\theta - i\pi\beta) F_\alpha(\theta + i\pi\beta) &= F_{\alpha-\beta}(\theta) F_{\alpha+\beta}(\theta)\,,\\
F_0(\theta) &\equiv 1\,.
\end{aligned}
\tag{11}
$$

When $0 < \alpha < \frac{1}{2}$, $F_\alpha(\theta)$ has simple poles at $i\alpha\pi$ and $i(1-\alpha)\pi$ of residues $2i\tan\alpha\pi$ and $-2i\tan\alpha\pi$, respectively, as well as zeros at $-i\alpha\pi$ and $-i(1-\alpha)\pi$. $F_{1/2}(\theta)$ has a double pole at $i\pi/2$ and a double zero at $-i\pi/2$.

The poles of a purely elastic S-matrix (paired with zeros via the unitarity condition (4)) encapsulate the dynamics of the theory. More specifically, the poles of each S-matrix element in the strip $-\pi < \mathrm{Im}\,\theta \leqslant \pi$ specify uniquely up to a sign the building blocks $f_\alpha(\theta)$ into which this S-matrix element factorizes. An overall minus sign, which is significant as will be seen later on, is equivalent to an extra factor of $f_1(\theta)$. In the S-matrix theories that are of interest to us here (see sect. 3), all simple poles in the physical strip correspond to bound states*, there are no "redundant" poles [32]. The bound states corresponding to the poles of the amplitude $S_{ab}(\theta)$ propagate either in the direct channel or in one of the two crossed channels, which are the direct channels of the (equal) amplitudes $S_{a\bar{b}}(\theta)$ and $S_{\bar{a}b}(\theta)$. (Recall that if a or b is self-conjugate then all *three* amplitudes are equal.)

If $S_{ab}(\theta)$ has a simple pole at $\theta_{ab} = iu_{ab}^{c}$ in the *direct* channel, we say that the particle c of mass

$$m_c^2 = s(iu_{ab}^c) = m_a^2 + m_b^2 + 2m_a m_b \cos u_{ab}^c \tag{12}$$

is a bound state of a and b. (If there are other particles of the same mass as c, one needs other quantum numbers to uniquely identify the particle c.) The particle c can be identical to a or b, of course. Eq. (12) has a geometric interpretation in terms of a triangle with sides of lengths m_a, m_b and m_c, implying that

$$u_{ab}^c + u_{\bar{c}a}^{\bar{b}} + u_{b\bar{c}}^{\bar{a}} = 2\pi\,. \tag{13}$$

Here and in the following we use eq. (12) to define u_{ab}^c as a function of the masses m_a, m_b and m_c, whenever c is a bound state of a and b. We also define $\bar{u}_{ab}^c \equiv \pi - u_{ab}^c$. Whether a simple pole of $S_{ab}(\theta)$ corresponds to a particle c propagating in the direct or crossed channel, depends on the coupling g_{ab}^c between the (incoming) particles a, b and the (outgoing) particle c. To be precise, if the purely imaginary residue of the simple pole has the same (opposite) sign as $i(g_{ab}^c)^2$, then the particle c occurs in the direct (crossed) channel; this may sound like a rather tautological statement, because in S-matrix theory the coupling g_{ab}^c itself is defined (up to a sign) by saying that the residue of $S_{ab}(\theta)$ at the simple pole at $\theta_{ab} = iu_{ab}^c$ is $i(g_{ab}^c)^2$ (see, however, below). But this statement is *not* empty, for a very simple reason. Namely, given the masses of a and b, different choices for the

* Actually, the simple poles in the S-matrix theories with Z-factors (see below) should not literally be referred to as bound state poles: In contrast to true bound states in, say, the sine-Gordon model, the masses of the particles corresponding to these simple poles do not depend on the coupling; in particular, these particles do not disappear as the coupling goes to zero. Furthermore – again different from true bound state poles – these simple poles can be seen in finite orders of perturbation theory. Finally, the purely elastic scattering of particles in the minimal S-matrix theories leads to large time delays (large compared to the time it would take a free particle to traverse the interaction region) for relative rapidities up to O(1). This is characteristic of the formation of quasi-stationary states, consistent with the existence of bound states in the minimal S-matrices. These time delays are absent in the S-matrices with Z-factors (this follows from the S-matrices presented in sect. 3), indicating that there are no bound states. Having said this, we will nevertheless (following other recent papers) refer to the simple poles in the S-matrices with Z-factors as bound states, since they satisfy all the properties of true bound states that will be relevant in what follows. For instance, scattering amplitudes factorize on any simple pole, which is important for the bootstrap equation (15).

sign of $(g^c_{ab})^2$ correspond to different masses for the particle c. If we choose $(g^c_{ab})^2$ such that c occurs as a simple pole at iu^c_{ab} in the direct channel of $S_{ab}(\theta)$, its mass is $m_c^2 = s(iu^c_{ab})$, as mentioned above. On the other hand, suppose we choose the opposite sign for $(g^c_{ab})^2$. Then c occurs in the crossed channel of $S_{ab}(\theta)$, i.e. c is a bound state of a and $\bar{b}$ (or $\bar{a}$ and b), since by crossing symmetry, eq. (6), it appears as a simple pole at $i\bar{u}^c_{ab}$ in the direct channel of $S_{ab}(\theta) = S_{\bar{a}b}(\theta)$. The mass of the particle c is then $m_c^2 = s(i\bar{u}^c_{ab})$. So we see that in any given theory with known masses the signs of all $(g^c_{ab})^2$ are determined by self-consistency. Let us also note that when constructing a diagonal S-matrix theory using the bootstrap, a procedure to be explained below, one has to make a choice for the sign of each new independent coupling $(g^c_{ab})^2$ encountered during the bootstrapping process. Not any new coupling encountered is independent; some couplings are related because we require the three-incoming-particle couplings $g_{abc} \equiv g^{\bar{c}}_{ab}$ to be totally symmetric in a, b and c, and to have the charge-conjugation symmetry $g_{\bar{a}\bar{b}\bar{c}} = g_{abc}$.

The above rule for assigning simple poles to the different channels is consistent with – and follows from – the total symmetry of the g_{abc} and the bootstrap, eq. (15) below, only if $S_{aa}(0) = -1$ for all particles a in the model. The latter condition, is indeed satisfied in all the algebra-related theories of sect. 3. Otherwise, as is the case for the sine-Gordon model at the reflectionless points (see end of subsect. 3.3), the rule has to be modified, taking into account the nontrivial parities and statistics of the particles [33, 34].

An S-matrix theory in which all couplings g^c_{ab} are real is called *one-particle unitary*. There are diagonal S-matrix theories, to be described in subsect. 3.2, violating one-particle unitarity; some of their couplings are purely imaginary. An S-matrix theory violating one-particle unitarity still has a unitary S-matrix in the sense of eq. (4); the unitarity of an S-matrix is just a consequence of the completeness of states in a theory [3]. One might naively assume that an S-matrix theory violating one-particle unitarity cannot correspond to a unitary QFT. However, Smirnov [23] has recently suggested that the S-matrices of subsect. 3.2 do actually describe unitary QFTs – although the unitarity of these theories arises in a somewhat subtle way; more on this in subsect. 3.2.

The amplitudes $S_{ab}(\theta)$ may also have higher-order poles, which are interpreted [5, 35] as arising from secondary scattering processes of the "constituents" of the particles a, b. These "constituents" must belong to the spectrum of the theory, and the secondary scattering processes must be allowed by the three-particle couplings as determined from the simple poles of the S-matrix; otherwise, the S-matrix cannot be regarded as complete and one has to specify the scattering amplitudes for the new "constituent" particles with all the particles already known to appear in the model. These considerations put rigid constraints on the position of multiple poles; for details, see refs. [5, 35]. For the S-matrices conjectured to be those of affine Toda field theories it was suggested in ref. [5] that *all* odd-order poles in the physical strip, not only simple ones, should be interpreted as bound states would be absolutely crucial if in some cases this would force one to introduce new bound states – not seen as simple poles in any S-matrix element – into the theory. If one considers all higher odd-order poles as corresponding to bound states, this is fortunately not the case in any of the S-matrix theories known to us.

We suggest that the general rule for when a multiple pole of $S_{ab}(\theta)$ should be considered as a bound-state pole is simply that it should be a simple pole of the secondary scattering of some of the "constituents" of a and b. (Note that then higher-order poles can *never* introduce bound states into the theory which are not also seen as simple poles in some S-matrix element.) It is not implausible, but as far as we know has not been proven, that a simple pole of the scattering of "constituents" always corresponds to a higher odd-order pole of $S_{ab}(\theta)$, and vice versa.

Finally, zeros in the physical strip, which by unitarity (4) are accompanied by poles outside the physical strip (on the "second sheet" in terms of the variable s), do not correspond to bound states.

For the S-matrix theories considered in this paper we will assume that higher odd-order poles correspond to bound states. The only reason we have to make such a decision, is that we would like to introduce the notion of *massive fusion rules* (called "bootstrap fusions" in ref. [7]). This notion is useful in the discussion of symmetries of S-matrix theories. We define the massive fusion rule coefficients M_{ab}^{c} by

$$M_{ab}^{c} = \begin{cases} 1, & \text{if } c \text{ is a bound state of } a \text{ and } b\,, \\ 0, & \text{otherwise}\,. \end{cases} \tag{14}$$

The total symmetry of the couplings g_{abc} implies the total symmetry of the $M_{abc} \equiv M_{ab}^{\bar{c}}$. Also $M_{\bar{a}\bar{b}\bar{c}} = M_{abc}$. If the S-matrix theory has a nontrivial global symmetry, expressed by the fact that the particles carry some discrete charge, then the massive fusion rules must be consistent with the conservation of this charge; i.e. M_{ab}^{c} may be nonzero only if the (additive, say) charge of the particle c equals the sum of the charges of the particles a and b.

So far we discussed the properties of purely elastic S-matrix theories. We now turn to their construction. Factorizable scattering theories in $1+1$ dimensions can be constructed using the *bootstrap principle* [2, 33, 34, 36]. The basic idea is that starting from a small number – one or two in the cases we will consider – of "fundamental" particles whose scattering amplitudes are assumed to be known, one can construct the scattering amplitudes of bound states formed by the fundamental particles using the "fusion procedure". This procedure is then applied again to the bound states formed out of the bound states, and so on. Different "fusion paths" leading to the same bound state are required, of course, to give rise to the same scattering amplitudes for this bound state. If one is fortunate, the bootstrap closes on a finite number of particles; that is, the bound state poles in the scattering amplitudes of all particles correspond to exactly the same particles, and no others*. (As previously mentioned, the closure of the known diagonal S-matrix theories is independent of counting only simple poles or all higher odd-order poles as corresponding to bound states.) In accordance with Chew's idea [37] of "nuclear democracy", the fundamental particle(s) is (are) not unique. Starting the bootstrap with other fundamental particles can give rise to the same model, in which particles which before were regarded as fundamental are now bound states of the new fundamental particles. Nevertheless, in all cases known to us, there is a set of fundamental particles whose scattering amplitudes are simpler (in particular they do not have any multiple poles) than those of any of the other sets of fundamental particles giving rise to the same model. The particles in this set will be referred to as *the fundamental particles***.

Now a few more details. Consider a particle c appearing as a direct-channel bound state in the scattering of particles a and b, corresponding to a simple pole at the relative rapidity $\theta_{ab} = iu_{ab}^{c}$. One then formally *defines* the bound state $|c(\theta)\rangle$ at $\theta = \theta_a + \theta_b$ as the projection on the pole of the two-particle state $|a(\theta_a)b(\theta_b)\rangle$ at the relative rapidity $\theta_{ab} = iu_{ab}^{c}$ [34]. In other words, the scattering amplitude of the particle c with any other particle d is by *definition* given by the residue (divided by $i(g_{ab}^{c})^2$) of the pole of the scattering amplitude of a, b and d, $S_{abd}(\theta_a, \theta_b, \theta_d) = S_{ab}(\theta_{ab})S_{ad}(\theta_{ad})S_{bd}(\theta_{bd})$, at $\theta_{ab} = iu_{ab}^{c}$. This gives the bootstrap equation [2]

$$S_{cd}(\theta) = S_{ad}(\theta + i\bar{u}_{a\bar{c}}^{\bar{b}})S_{bd}(\theta - i\bar{u}_{b\bar{c}}^{\bar{a}}) . \tag{15}$$

This equation, embodying the fusion procedure, should hold for all particles a, b and c such that $g_{ab}^{c} \neq 0$. As we already mentioned, given that $S_{aa}(0) = S_{bb}(0) = -1$, it follows from the bootstrap equation that $-i \operatorname{Res} S_{ab}(\theta)_{|\theta = iu_{ab}^{\bar{c}}}$ is totally symmetric in a, b and c, and therefore can be consistently identified as $(g_{ab\bar{c}})^2$.

Related to the fusion procedure are constraints on purely elastic scattering theories arising from the existence of nontrivial IMs. If these IMs are local and diagonal on asymptotic one-particle states, Lorentz covariance requires that a conserved charge Q_s of Lorentz spin s act on an N-particle (asymptotic) state in the momentum basis as follows:

$$Q_s \big| a_1(\theta_1) \dots a_N(\theta_N) \big\rangle = \sum_{i=1}^{N} \gamma_s^{a_i} \, e^{s\theta_i} \big| a_1(\theta_1) \dots (\theta_N) \big\rangle , \tag{16}$$

* A priori it is not obvious to us if a model arising from a bootstrap closing on *infinitely* many particles is always unphysical for some reason, or not. In any case, no bootstrap closing on infinitely many particles seems to be known.

** If *the* fundamental particle(s) is (are) the lightest particle(s) in the model, it is easy to see that kinematics rules out secondary scattering of its (their) "constituents", explaining the absence of multiple poles in this case.

where $\gamma_s^{a_i}$ are some (real) coefficients. This simply follows from the fact that the only single-particle variables of Lorentz spin 1 in $1+1$ dimensions are the left light-cone momenta $p_i = p_i^0 + p_i^1 = m_i\,\mathrm{e}^{\theta_i}$. (Note that the right light-cone momenta $\bar{p}_i = p_i^0 - p_i^1 = m_i\,\mathrm{e}^{-\theta_i}$ have Lorentz spin -1.) Parity invariance implies that for each IM Q_s there is an IM Q_{-s} with the opposite Lorentz spin $-s$ and the same coefficients $\gamma_{-s}^a = \gamma_s^a$. We can therefore restrict ourselves to positive s, for simplicity. As there is no true spin in $1+1$ dimensions, we will also just write "spin" when we mean "Lorentz spin".

When we require eq. (16) to hold for complex rapidities, the previously mentioned definition of a bound state c of two particles a and b leads to the following condition on the coefficients γ_s^a:

$$\gamma_s^a\,\mathrm{e}^{-is\bar{u}_{ac}^{\bar{b}}} + \gamma_s^b\,\mathrm{e}^{is\bar{u}_{bc}^{\bar{a}}} = \gamma_s^c\,. \tag{17}$$

The condition that this equation holds for all a, b, c such that $g_{ab}^c \neq 0$, and $\gamma_s^a \neq 0$ for at least one a, is a necessary and sufficient condition for the existence of a local IM of spin s. If all the angles u_{ab}^c are known, one can solve eq. (17) to find the spins s of the IMs and the explicit values of the γ_s^a for $s>0$. Note that for $s=1$ the γ_s^a are, up to an overall factor which we can choose to be 1, just the masses m_a of the particles in the theory. A concise way of stating the values of all the γ_s^a for the Lie algebra-related S-matrix theories discussed in this paper will be given in subsect. 3.7.

The application of the bootstrap in practice is not always straightforward. One first of all determines the *minimal* S-matrix of a theory. This is the S-matrix whose fundamental amplitude(s) – and therefore all amplitudes – have the smallest number of poles and zeros in the physical strip, among all those S-matrix theories giving rise to the same mass spectrum and massive fusion rules. The point is that one can usually multiply any given fundamental amplitude $S_{\mathrm{ff'}}(\theta)$ (f and f′ for fundamental) by some suitable product $Z_{\mathrm{ff'}}(\theta)$ of $f_\alpha(\theta)$ with negative α, such that "bootstrapping" starting from the new fundamental amplitude(s) $Z_{\mathrm{ff'}}(\theta)S_{\mathrm{ff'}}(\theta)$ will produce an S-matrix which still satisfies the constraints of unitarity and crossing, and has the same mass spectrum and massive fusion rules. The $Z_{ab}(\theta)$ are the so-called Z-factors.

The possibility of introducing Z-factors is advantageous when considering the affine Toda field theories. Since the mass spectrum of these theories does not depend on their coupling β (at least in the simply laced case), it is natural to conjecture that each of them has a fixed minimal S-matrix, encoding the mass spectrum, and the dependence on the coupling enters through the Z-factors. We will see in sect. 3 that in the cases under consideration, there are Z-factors that actually depend on an arbitrary (within a certain region) parameter which will be denoted by b. As b tends to 0, $Z_{cd}(\theta)S_{cd}(\theta)$ goes to 1 for all c and d, corresponding to a free theory. For the ATFTs b must therefore be a function of β which tends to 0 as $\beta \to 0$.

In the determination of the fundamental amplitudes – even the minimal ones – some guesswork may be needed. In the case of affine Toda field theories things are straightforward. As remarked in sect. 1, ATFTs based on simply laced algebras seem to be integrable at the quantum level, their mass ratios do not renormalize, and three-particle couplings which are zero or nonzero in the tree-level lagrangian also seem to stay zero or nonzero, respectively, at the quantum level [16]. Granted the above, if the masses are all different, the S-matrix of an ATFT must obviously be purely elastic. In general, if there are mass degeneracies in a factorizable theory, there can be amplitudes "mixing" different particles of the same mass, and even if the in- and out-particles are the same in any scattering process, there can be reflection amplitudes. However, if there are enough IMs of spin $s > 0$ such that their "eigenvalues" γ_s^a of eq. (16) allow one to distinguish between all particles, clearly no "mixing" of particles is possible, and it is also easy to see [6] that then the reflection amplitude of any two particles must vanish. In the case of an ATFT based on a simply laced affine Lie algebra, one is in the fortunate situation (again granted the above assumptions) of knowing the masses and which couplings are nonzero from the tree-level lagrangian, i.e. without having to know the exact S-matrix of the theory. So it is possible to calculate all γ_s^a from eq. (17) and see that indeed there is always an IM of spin $s > 0$ which distinguishes particles of equal mass. (We will find in subsect. 3.7 that this is the case for all the Lie algebra-related theories discussed in this paper, although without using the ATFT lagrangians.) Hence the S-matrix is diagonal, and the fundamental amplitudes are just the products of $f_\alpha(\theta)$ which have poles corresponding to the particles that couple to the fundamental particles (usually there is a natural choice for the fundamental particles, e.g. the lightest particle if all particle masses are different). On the other hand, if one is interested in a perturbed CFT, all one knows are the spins of the IMs and the global symmetry of the model, which is just that of the CFT not broken by the perturbing field. It then requires a little bit more ingenuity to find an ansatz for the fundamental scattering amplitudes. The main help comes from eq. (17), which, since the spins of the IMs are known, allows one to determine some of the mass ratios in the theory. For examples of how one proceeds in such a case, we refer the reader to refs. [2, 6, 7].

3. Purely elastic S-matrices

In this section we describe the purely elastic S-matrix theories related to affine Lie algebras that have recently been discussed in connection with integrable perturbations of CFTs. In each case we give the mass spectrum, the minimal two-particle scattering amplitudes (or at least the fundamental ones, from which all the other amplitudes can be obtained using the bootstrap equations and

$$f_\alpha(\theta) = \frac{\sinh\frac{1}{2}(\theta + i\alpha\pi)}{\sinh\frac{1}{2}(\theta - i\alpha\pi)} \;;\quad f_1(\theta) = \frac{\sinh\frac{1}{2}(\theta + i\pi)}{\sinh\frac{1}{2}(\theta - i\pi)} = -1$$

crossing symmetry), and the corresponding Z-factors. We also discuss the symmetry of each S-matrix theory and the perturbed CFT which the minimal S-matrix is proposed to describe. At the end of the section we discuss features that are common to all the theories.

3.1. $A_n^{(1)}$ $(n \geqslant 1)$

The n particles in the model will be labeled by $a = 1, 2, \ldots, n$, where $\bar{a} = n + 1 - a$. Normalizing the mass of the fundamental particle 1 to unity (as we will do in all the theories), the masses are

$$m_a = \sin\left(\frac{\pi a}{n+1}\right) \Big/ \sin\left(\frac{\pi}{n+1}\right). \tag{18}$$

The fundamental S-matrix element is [38]

$$S_{11}(\theta) = f_{2/(n+1)}(\theta). \tag{19}$$

The full two-particle S-matrix, obtained by "bootstrapping" S_{11}, can be written as

$$S_{ab}(\theta) = f_{|a-b|/(n+1)}(\theta)\left[\prod_{k=1}^{\min(a,b)-1} f_{(|a-b|+2k)/(n+1)}(\theta)\right]^2 f_{a+b/(n+1)}(\theta). \tag{20}$$

where $a, b = 1, 2, \ldots, n$. Note that when $a + b > n + 1$, there are cancellations that reduce the number of factors in the expression. The fundamental Z-factor is [21]

$$Z_{11}(\theta) = f_{-b}(\theta) f_{-2/(n+1)+b}(\theta). \tag{21}$$

For $n > 1$, the model exhibits a $Z_2 \ltimes Z_{n+1}$ symmetry. Each particle is assigned a Z_{n+1}-charge [38], the charge of the particle a is just a, which is conserved by the massive fusion rules of the model. The latter are given by $M_{ab}^c = 1$ if and only if $a + b \equiv c \pmod{n+1}$. The additional Z_2 symmetry is charge conjugation, exchanging particles with their antiparticles.

A.B. Zamolodchikov has proposed [1] that the above minimal solution in the case $n = 2$, describes the perturbation of the CFT associated to the critical three-state Potts model ($c = \frac{4}{5}$) by the energy density operator $\phi_{(2,1)}$ of weight $(\frac{2}{5}, \frac{2}{5})$. The natural generalization to $n > 2$ is (cf. refs. [6, 8, 16, 39]) that the above nth minimal S-matrix theory describes the perturbation of the first model in the unitary series of the $W(A_n)$-algebra [14] ($c = 2n/(n+3)$) by the primary field of weight $(2/(n+3), 2/(n+3))$. This CFT is also known [14] to describe Z_{n+1} parafermions [40], and the perturbing field preserves the global symmetry. Finally, the above discussion applies to the case $n = 1$ as well, where one gets $S_{11}(\theta) \equiv -1$ as the full S-matrix for the thermal perturbation of the Ising model [38] (the "Ising field theory", obtained by taking a massive scaling limit of the Ising model from above the critical temperature), with the single particle in the model corresponding to the bosonic order variable σ of the Ising model.

3.2. $A^{(2)}_{2n}$ ($n \geqslant 1$)

The masses of the n self-conjugate particles $a = 1, 2, \ldots, n$ in the theory are

$$m_a = \sin\left(\frac{\pi a}{2n+1}\right) \Big/ \sin\left(\frac{\pi}{2n+1}\right). \tag{22}$$

The general minimal two-particle amplitude is [4, 23]

$$S_{ab}(\theta) = F_{|a-b|/(2n+1)}(\theta) \left[\prod_{k=1}^{\min(a,b)-1} F_{(|a-b|+2k)/(2n+1)}(\theta) \right]^2 F_{(a+b)/(2n+1)}(\theta). \tag{23}$$

(The $n = 1$ case was treated in refs. [5, 21].) The fundamental Z-factor (given in ref. [21] for $n = 1$) is

$$Z_{11}(\theta) = F_{-b}(\theta) F_{-2/(2n+1)+b}(\theta). \tag{24}$$

Comparing eq. (23) with eq. (20), we observe (cf. ref. [27]) that for $a, b = 1, 2, \ldots, n$

$$S^{A^{(2)}_{2n}}_{ab}(\theta) = S^{A^{(1)}_{2n}}_{ab}(\theta) S^{A^{(1)}_{2n}}_{ab}(i\pi - \theta). \tag{25}$$

This "folding" of the $A^{(1)}_{2n}$ S-matrix theory into the $A^{(2)}_{2n}$ theory has interesting consequences that will be discussed in sect. 5. The minimal $A^{(2)}_{2n}$ theories are the only ones discussed in the present paper that violate one-particle unitarity, having some purely imaginary couplings [3, 4]. As mentioned in sect. 2, Smirnov [23] has suggested that these S-matrix theories nevertheless correspond to unitary massive QFTs, related to "restricted sine-Gordon models". By examining form factors of local fields he argues that one can eliminate the solitons from the spectrum of the sine-Gordon model at some special values of the coupling ($\frac{1}{2}\beta^2 = 2/(2n+3)$, in a normalization where the coupling is in the range $0 \leqslant \beta^2 \leqslant 2$), while still preserving the locality and unitarity of the theory. The above S-matrices are indeed just those of the "breathers" in the sine-Gordon model at these special values of the coupling.

Note that the Z-factors change the sign of certain simple-pole residues, exactly restoring one-particle unitarity in the nonminimal models. Neither the minimal nor the nonminimal theories have any nontrivial discrete symmetry. The fusion rule coefficients are $M^c_{ab} = 1$ for $c = \min(a+b, 2n+1-a-b)$ and also for $c = |a-b|$ when $a \neq b$.

The minimal $A_{2n}^{(2)}$-related S-matrix theories are proposed [3,4,23] to describe the $\phi_{(1,3)}$ perturbation [weight $(-(2n-1)/(2n+3), -(2n-1)/(2n+3))$] of the nonunitary Virasoro minimal models labeled by $p'=2$, $p=2n+3$ of central charge $c=-2n(6n+5)/(2n+3)$.

3.3. $D_n^{(1)}$ $(n \geqslant 2)$

For these models* we have to distinguish between two cases, depending on whether n is even or odd.

3.3.1. $D_n^{(1)}$, n even. The n particles, all self-conjugate, will be labeled by $1,2,\dots,n-2$, f_1, and f_2. Their masses are

$$m_{f_1}=m_{f_2}=1, \qquad m_a = 2\sin\left(\frac{\pi a}{2(n-1)}\right) \qquad \text{for } a=1,2,\dots,n-2. \tag{26}$$

The complete two-particle S-matrix can be determined, via the bootstrap, from the following three fundamental amplitudes (the case $n=4$ was discussed explicitly in ref. [16]):

$$S_{f_1f_1}(\theta)=S_{f_2f_2}(\theta)=-S_{f_1f_2}(\theta)=-\prod_{k=1}^{n-2} f_{k/(n-1)}(\theta)=(-1)^{n/2}\prod_{k=1}^{\frac{1}{2}n-1} F_{2k/(n-1)}(\theta). \tag{27}$$

The Z-factors are

$$Z_{f_1f_1}(\theta)=Z_{f_2f_2}(\theta)=(-1)^{n/2}\prod_{k=0}^{\frac{1}{2}n-1} F_{-k/(n-1)-(-1)^k b}(\theta),$$

$$Z_{f_1f_2}(\theta)=-(-1)^{n/2}\prod_{k=1}^{\frac{1}{2}n-1} F_{-k/(n-1)+(-1)^k b}(\theta). \tag{28}$$

For the reader's convenience we give here the rest of the minimal two-particle amplitudes:

$$S_{af_1}(\theta)=S_{af_2}(\theta)=(-1)^a\prod_{k=0}^{a-1} F_{\frac{1}{2}-(a-2k)/2(n-1)}(\theta) \qquad \text{for } a=1,2,\dots,n-2, \tag{29}$$

and $S_{ab}(\theta)$, where $a,b=1,2,\dots,n-2$, is simply given by eq. (23) with every "$2n+1$" there replaced by "$2(n-1)$". Note that, due to the symmetry $F_a(\theta)=F_{1-a}(\theta)$, there are third- and fourth-order poles in amplitudes where $a+b \geqslant n$,

* The following $D_n^{(1)}$-related S-matrices have recently also been presented by Braden et al. [20].

poles which are absent in the $A_{2n}^{(2)}$ amplitudes. The third-order poles imply $M_{ab}^{2(n-1)-a-b}=1$, for $a+b \geqslant n$. The other nonzero fusion coefficients can be deduced from $M_{f_1 f_1}^{a_{\rm even}} = M_{f_2 f_2}^{a_{\rm even}} = M_{f_1 f_2}^{a_{\rm odd}} = 1$, $M_{ab}^{|a-b|}=1$ for $a \neq b$, $M_{ab}^{a+b}=1$ for $a+b \leqslant n-2$, using the symmetries discussed in sect. 2. For $n \geqslant 6$ the model is $Z_2 \ltimes (Z_2 \times Z_2)$ invariant. The $Z_2 \times Z_2$ charges of the particles $1,2,3,\dots,n-2$, f_1, f_2 are $(1,1),(0,0),(1,1),\dots,(0,0),(1,0),(0,1)$, respectively. The additional Z_2 interchanges f_1 and f_2. For $n=4$ this latter Z_2 is replaced by S_3, enlarging the symmetry (see further discussion below). For $n=2$ the S-matrix becomes that of two decoupled $A_1^{(1)}$-related models, as expected from $D_2 = A_1 \oplus A_1$.

3.3.2. $D_n^{(1)}$, *n* odd. In this case we label the n particles by $1,2,\dots,n-2$, f, and $\bar f$. The first $n-2$ are self-conjugate and the last two are conjugate to one another. Their masses are

$$m_f = m_{\bar f} = 1, \qquad m_a = 2\sin\left(\frac{\pi a}{2(n-1)}\right), \quad \text{for } a = 1,2,\dots,n-2\,. \tag{30}$$

Here it is sufficient to specify only one amplitude, either S_{ff} or its crossing-symmetric partner $S_{f\bar f}$ which are given by

$$S_{ff}(\theta) = -S_{f\bar f}(\theta) = \prod_{k=1}^{n-2} f_{k/(n-1)}(\theta)\,. \tag{31}$$

The corresponding Z-factors are

$$Z_{ff}(\theta) = \prod_{k=0}^{n-2} f_{-k/(n-1)-(-1)^k b}(\theta), \qquad Z_{f\bar f}(\theta) = \prod_{k=1}^{n-1} f_{-k/(n-1)+(-1)^k b}(\theta)\,. \tag{32}$$

The other minimal amplitudes, involving the particles $1,2,\dots,n-2$, take the same form as in the n even case. The symmetry of the model is $Z_2 \ltimes Z_4$, unlike in the n even case. The particles $1,2,3,\dots,n-2$, f, and $\bar f$ are assigned the Z_4-charges $2,0,2,\dots,2,1,3$, respectively. The extra Z_2 symmetry corresponds to charge conjugation. The nonzero fusion rule coefficients are $M_{ff}^{a_{\rm odd}}$, $M_{f\bar f}^{a_{\rm even}}$, $M_{ab}^{|a-b|}$ for $a \neq b$, M_{ab}^{a+b} for $a+b \leqslant n-2$, $M_{ab}^{2(n-1)-a-b}$ for $a+b \geqslant n$, and the ones obtained from the above by the general symmetries of M_{ab}^c. Note that in the $n=3$ case the S-matrix theory is identical to the $A_3^{(1)}$-related one, which is to be expected from the equivalence of the corresponding algebras.

In both cases, n even or odd, the $D_n^{(1)}$-related S-matrix theories are conjectured [6] to describe the perturbation of the first models in the $W(D_n)$ unitary series by the weight $(1/n, 1/n)$ primary field. The global symmetry common to all the models in these series [14] – $Z_2 \times Z_2$ (Z_4) for even (odd) n – is enlarged in their first models by an additional Z_2 (S_3 for $n=4$), and the perturbing field does not destroy it. These CFTs are also identified [14] with special points on the $c=1$

critical Ashkin–Teller line, with the orbifold compactification radius [41] being $r_{\mathrm{orb}} = \sqrt{n/2}$. In particular, the $n=4$ case corresponds to the four-state Potts model whose enhanced S_4 symmetry is manifest also in the $D_4^{(1)}$ S-matrix theory: The mass-degenerate particles f_1, f_2, and m_1 form a three-dimensional irreducible representation of S_4. As noted in ref. [16], the triple mass degeneracy that occurs in other $D_{3k+1}^{(1)}$ related theories do not yield an irreducible triplet of particles. Other particular cases are $n=3$, where the special orbifold point corresponds to Z_4-parafermions (in agreement with the algebra equivalence $D_3 = A_3$ mentioned above), and $n=2$, where $r_{\mathrm{orb}} = 1$ gives the (Ising)^2 point (in accordance with $D_2 = A_1 \oplus A_1$).

We note a suggestive correspondence between the massive fusion rules and symmetries of the $D_n^{(1)}$-related S-matrix theories and those of the operator product algebras of the unperturbed CFTs (the latter were studied in detail by Dijkgraaf et al. [42]; we adopt here their notation, their "N" identified with our "n"): The fundamental particles of the S-matrix theory can be thought of as counterparts of the twist fields $\sigma_{1,2}$ of the CFT (in particular, the conjugation properties of the latter are the same as those of the fundamental particles, i.e. different for n even or odd), while the other particles $a = 1,2,\ldots,n-2$ in the massive theory correspond to the first $n-2$ vertex operators ϕ_a in the CFT. The perturbing field in this notation is ϕ_2.

As mentioned in ref. [16], the minimal $D_n^{(1)}$-related S-matrices closely resemble those of the sine-Gordon model at the reflectionless points where the $(n-1)$th breather is at threshold. We note that although the masses in these theories are identical, there are sign differences in amplitudes involving the fundamental particles. In the sine-Gordon model the latter are the soliton s and the antisoliton $\bar{s}$, and the amplitudes in question are [33]

$$S_{s\bar{s}}(\theta) = (-1)^n S_{ss}(\theta) = \prod_{k=1}^{n-2} f_{k/(n-1)}(\theta)\,,$$

for n=2Z+1 coincides with (31)

$$S_{as}(\theta) = S_{a\bar{s}}(\theta) = \prod_{k=0}^{a-1} F_{\frac{1}{2}-(a-2k)/2(n-1)}(\theta) \qquad \text{for } a = 1,2,\ldots,n-2. \quad (33)$$

$(-1)^a$

Theories with different S-matrices – even if differing only by signs – must come from different QFTs*. Indeed, we can show more explicitly that the reflectionless sine-Gordon models are different from the minimal $D_n^{(1)}$ models. First of all, the sign changes in some scattering amplitudes lead to different massive fusion rules for the sine-Gordon models; now *all* the particles $a = 1,2,\ldots,n-2$ (the breathers)

* In fact, the off-shell behavior can be very different, as the example of the Ising field theory shows. Its almost trivial S-matrix $S \equiv -1$ is "observationally" indistinguishable (see below) from that of a free particle, $S \equiv 1$, but the Green functions of the Ising field theory are of course highly nontrivial [43].

are bound states of s and $\bar{s}$, and there are no bound states in the direct channels of the amplitudes S_{ss} and $S_{\bar{s}\bar{s}}$. These fusion rules reflect the conservation of the topological charge in the sine-Gordon model (the U(1) charge, in the massive Thirring model language): The soliton and antisoliton are oppositely charged whereas all the breathers are neutral. The U(1) symmetry is also present in the unperturbed CFTs – massless Thirring models, which are $c = 1$ gaussian theories. These CFTs are different from the ones whose perturbations lead to the $D_n^{(1)}$-related models above, as we said, the latter CFTs are $c = 1$ orbifold models, in which the U(1) symmetry is broken. We note in particular the $n = 2$ case, where the reflectionless sine-Gordon model is equivalent to the *free* massive Thirring model (a free massive Dirac fermion), and the corresponding unperturbed CFT is the Dirac Point [41] $r_{\text{circle}} = 1$ on the gaussian line, as opposed to the $(\text{Ising})^2$ CFT whose perturbation is described by the $D_2^{(1)}$ scattering theory. The sign differences in the scattering amplitudes will also turn out to be significant later on (see subsect. 5.1.3). Finally, we remark that we were not able to find nontrivial Z-factors that could be added to the reflectionless sine-Gordon S-matrices (the coupling constant is already fixed!).

At this point we would like to comment on some amusing features of scattering theory in 1 + 1 dimensions and purely elastic scattering in particular, which actually will turn out to be important later. The only "observables" in a purely elastic scattering theory are the time delays (compared to the free case) in the scattering of any two particles described by wave packets. In the scattering of particles a and b, the time delays of a and b depend on the S-matrix only through the rapidity derivative of the phase shift, $\varphi_{ab}(\theta) = -i(\mathrm{d}/\mathrm{d}\theta)\ln S_{ab}(\theta)$. In particular, purely elastic scattering theories whose scattering matrix elements differ only by signs, like the minimal $D_n^{(1)}$ models and the reflectionless sine-Gordon points, are "observationally" indistinguishable. Note especially that one cannot distinguish the solitons in the reflectionless sine-Gordon models, which are fermions, from *the* fundamental particles in the $D_n^{(1)}$ models, which apparently are bosons (as will be argued in sect. 5). The reason is that the concept of *statistics of a particle* cannot be defined on a pure S-matrix level in two space-time dimensions. In higher dimensions one can distinguish bosons and fermions by the different angular dependence of their scattering amplitudes. But for scattering on a line this is of course not possible. Even at the quantum field-theoretic level there are some subtleties. First of all, there is no spin-statistics connection in (1 + 1)-dimensional QFT, for the excellent reason that there is no spin in one space dimension*. Furthermore, although the behaviour of wave functions under interchange of particles is well defined even for particles "living" on a line (and we use it to define the notions of fermions and bosons), it is physically rather irrelevant. For instance, we will see in

* Although there is a connection between *Lorentz spin* and statistics (commutation relations) for *fields*, see ref. [44] and references therein.

sect. 4 that particles which by this definition are called bosons can nevertheless obey an exclusion principle. This fact will lead us to the notion of the "type" of a particle [27], to be defined in sect. 4. In certain respects this notion replaces that of fermions and bosons in $1+1$ dimensions.

3.4. $E_6^{(1)}$

This model contains 6 particles which we label by $1, \bar{1}, 2, 3, \bar{3}$, and 4, according to increasing mass. In a convenient normalization these masses are

$$m_1 = m_{\bar{1}} = 1, \qquad m_2 = \sqrt{2}\,,$$
$$m_3 = m_{\bar{3}} = (1+\sqrt{3})/\sqrt{2}\,, \qquad m_4 = 1+\sqrt{3}\,. \tag{34}$$

The fundamental amplitude is [6, 7]

$$S_{11}(\theta) = f_{\frac{1}{6}}(\theta) f_{\frac{1}{2}}(\theta) f_{\frac{2}{3}}(\theta)\,, \tag{35}$$

and the corresponding Z-factor is

$$Z_{11}(\theta) = f_{-b}(\theta) f_{-\frac{1}{6}+b}(\theta) f_{-\frac{1}{2}-b}(\theta) f_{-\frac{2}{3}+b}(\theta)\,. \tag{36}$$

The full set of S-matrix elements (without Z-factors) is given in ref. [7]. The model exhibits a $Z_2 \ltimes Z_3$ symmetry, with the particles $1, \bar{1}, 2, 3, \bar{3}, 4$ carrying the Z_3-charges $1, 2, 0, 1, 2, 0$, respectively; the Z_2 exchanges particles with their antiparticles. This is also the global symmetry of the perturbed CFT that this S-matrix theory presumably describes [6, 7], namely the tricritical three-state Potts model ($c = \frac{6}{7}$) perturbed by the thermal operator $\phi_{(1,2)}$ of weight $(\frac{1}{7}, \frac{1}{7})$.

3.5. $E_7^{(1)}$

The seven self-conjugate particles in the model will be labeled by $a = 1, 2, \ldots, 7$, in order of increasing mass. Their masses, normalized so that $m_1 = 1$, are

$$m_2 = 2\cos(\tfrac{5}{18}\pi)\,, \qquad m_3 = 2\cos(\tfrac{1}{9}\pi)\,, \qquad m_4 = 2\cos(\tfrac{1}{18}\pi)\,,$$
$$m_5 = 4\cos(\tfrac{1}{18}\pi)\cos(\tfrac{5}{18}\pi)\,, \qquad m_6 = 4\cos(\tfrac{1}{9}\pi)\cos(\tfrac{2}{9}\pi)\,, \qquad m_7 = 4\cos(\tfrac{1}{18}\pi)\cos(\tfrac{1}{9}\pi)\,. \tag{37}$$

The minimal amplitude for the scattering of two lightest particles is [4–6]

$$S_{11}(\theta) = -F_{\frac{1}{9}}(\theta) F_{\frac{4}{9}}(\theta)\,, \tag{38}$$

and the corresponding Z-factor is [5]

$$Z_{11}(\theta) = -F_{-b}(\theta)F_{-\frac{1}{9}+b}(\theta)F_{-\frac{4}{9}-b}(\theta)\,. \tag{39}$$

The full S-matrix (including Z-factors) is given by Christe and Mussardo [5]. The symmetry of the theory is Z_2, with the particles $1,2,\ldots,7$ carrying the conserved charges $1,0,1,0,0,1,0$, respectively. The theory was proposed [2] to describe the energy perturbation (the primary field $\phi_{(1,2)}$ of weight $(\frac{1}{10},\frac{1}{10})$) of the tricritical Ising model ($c=\frac{7}{10}$).

3.6. $E_8^{(1)}$

We normalize the masses of the eight self-conjugate particles $a=1,2,\ldots,8$ in the model to

$$\begin{aligned} &m_1 = 1\,, && m_2 = 2\cos(\tfrac{1}{5}\pi)\,, && m_3 = 2\cos(\tfrac{1}{30}\pi)\,, \\ &m_4 = 4\cos(\tfrac{1}{5}\pi)\cos(\tfrac{7}{30}\pi)\,, && m_5 = 4\cos(\tfrac{1}{5}\pi)\cos(\tfrac{2}{15}\pi)\,, && m_6 = 4\cos(\tfrac{1}{5}\pi)\cos(\tfrac{1}{30}\pi)\,, \\ &m_7 = 8\cos^2(\tfrac{1}{5}\pi)\cos(\tfrac{7}{30}\pi)\,, && m_8 = 8\cos^2(\tfrac{1}{5}\pi)\cos(\tfrac{2}{15}\pi)\,. \end{aligned} \tag{40}$$

All the S-matrix elements can be obtained from the fundamental amplitude [2]

$$S_{11}(\theta) = F_{\frac{1}{15}}(\theta)F_{\frac{1}{3}}(\theta)F_{\frac{2}{5}}(\theta)\,, \tag{41}$$

and the corresponding Z-factor [16,45]

$$Z_{11}(\theta) = F_{-b}(\theta)F_{-\frac{1}{15}+b}(\theta)F_{-\frac{1}{3}-b}(\theta)F_{-\frac{2}{5}+b}(\theta)\,. \tag{42}$$

The theory does not have any nontrivial symmetry. The same is true for the perturbed CFT it is proposed to describe [2] namely the Ising model ($c=\frac{1}{2}$) perturbed by the weight $(\frac{1}{16},\frac{1}{16})$ magnetic operator $\phi_{(1,2)}$.

We remark that tables of the full S-matrices (including Z-factors) of the $E_6^{(1)}$-, $E_7^{(1)}$- and $E_8^{(1)}$-related theories have been published in ref. [20].

3.7. GENERAL COMMENTS ON THE ABOVE S-MATRIX THEORIES

We now make some comments on the above S-matrix theories, showing how many details of their structure are related to (affine) Lie algebras. The initial indication for such a relation was the fact that the spins of the IMs of perturbed CFTs, whose existence could be established using the counting argument, turned out to be the first few exponents of some affine Lie algebra $\hat{\mathscr{G}}$, modulo its Coxeter number $h_{\hat{\mathscr{G}}}$ (the latter periodicity can be recognized when $h_{\hat{\mathscr{G}}}$ is relatively small); in addition, the number of particles in the S-matrix theory related to this algebra is equal to the rank r of $\hat{\mathscr{G}}$ [2], and the smallest common denominator of the rational subscripts α of the basic building blocks $f_\alpha(\theta)$ appearing in the S-matrix is the Coxeter number $h_{\hat{\mathscr{G}}}$ ($\frac{1}{2}h_{\hat{\mathscr{G}}}$ for the $A_{2n}^{(2)}$-related models). For completeness, we give here the relevant affine Lie algebra data [12]:

$$
\begin{aligned}
&A_n^{(1)}: r=n, \quad h=n+1, \quad s=1,2,\ldots,n,\\
&A_{2n}^{(2)}: r=n, \quad h=2(2n+1), \quad s=1,3,5,\ldots,2n-1,2n+3,2n+5,\ldots,4n+1,\\
&D_n^{(1)}: r=n, \quad h=2(n-1), \quad s=1,3,5,\ldots,2n-3,n-1,\\
&E_6^{(1)}: r=6, \quad h=12, \quad s=1,4,5,7,8,11,\\
&E_7^{(1)}: r=7, \quad h=18, \quad s=1,5,7,9,11,13,17,\\
&E_8^{(1)}: r=8, \quad h=30, \quad s=1,7,11,13,17,19,23,29.
\end{aligned}
\tag{43}
$$

It was then realized [2,8,6] that the same algebra plays a role in one of the possible constructions of the unperturbed CFT: Except for $\hat{\mathscr{G}}=A_{2n}^{(2)}$, this CFT can be specified succinctly [6] as being the first model of the $W(\mathscr{G})$ unitary series [14]. Here $\mathscr{G}$ is the ordinary simple Lie algebra related to $\hat{\mathscr{G}}$. This construction of the CFT also corresponds to the coset $(\hat{\mathscr{G}}_1\times\hat{\mathscr{G}}_1)/\hat{\mathscr{G}}_2$. The $\hat{\mathscr{G}}$-related massive theory discussed earlier, is proposed to describe the perturbation of this CFT by the $W(\mathscr{G})$-primary field of weight $(2/(h_{\mathscr{G}}+2), 2/(h_{\mathscr{G}}+2))$. This primary field always preserves [8] the global symmetry common to the whole $W(\mathscr{G})$ unitary series [14], which is indeed always a part of the symmetry of corresponding S-matrix theory. However, in many cases the symmetry of the CFT is larger, as it can have alternative constructions, and the S-matrix theory should have this larger symmetry as well, if the perturbing field does not break it. To demonstrate this point, consider the $(Z_2\ltimes Z_3)$-symmetric tricritical three-state Potts model, which is a member of both the (generically Z_2-symmetric) $W(A_1)$ (Virasoro) and the (generically Z_3-symmetric) $W(E_6)$ unitary series. The weight $(\frac{1}{7},\frac{1}{7})$ perturbing field preserves both the Z_2 and the Z_3 symmetries, and this is consistent with the features of the $E_6^{(1)}$-related S-matrix theory of subsect. 3.4.

The $A_{2n}^{(2)}$-related S-matrix theories do not follow the above regularity. As already mentioned, they describe the perturbation of a family of nonunitary $W(A_1)$ (Virasoro) minimal models. The connection to a coset construction is presumably only a formal one, as indicated by the fact that the central charge of these CFTs equals that of the rational coset construction [46] $(A_{2n}^{(1)})_{-n-\frac{1}{2}}\times(A_{2n}^{(1)})_1/(A_{2n}^{(1)})_{-n+\frac{1}{2}}$.

Considering the S-matrix theories, it was further noticed [4, 16] that the masses in the $\hat{\mathscr{G}}$-related S-matrix theory are just (up to an overall factor) the components of the Perron–Frobenius vector of the incidence matrix $I_{\mathscr{G}}$*. The twisted affine algebra $A_{2n}^{(2)}$ does not have an associated ordinary Lie algebra (or rather, A_{2n} or A_n are not what we want); in this case the above statement still holds, if we use as a "generalized incidence matrix" the matrix obtained from I_{A_n} by replacing the 0 in the last entry along the diagonal by 1. This matrix will further show up in the analysis of the structure of the $A_{2n}^{(2)}$-related S-matrix theories (see below and in sect. 5). Parenthetically we remark, that for the ATFTs the masses arise as the square roots of the eigenvalues of a different matrix related to the root system of the affine $\hat{\mathscr{G}}$ [5, 16–18]. It is surprising, but true, that these two recipes for calculating the masses coincide for all Lie algebras, as has been shown by Braden [48].

It seems from the above discussion, that one could just as well argue for a relation between the S-matrix theories and *ordinary* simple Lie algebras: For an untwisted affine Lie algebra the exponents coincide with those of the associated ordinary Lie algebra, and the masses are related to the (Perron–Frobenius vector of the incidence matrix of the) ordinary Lie algebra anyhow. Another relation to Lie algebras, which can also be interpreted in two different ways, is seen in the discrete symmetries of the S-matrices. On the one hand, the symmetry of each of the algebra-related (minimal or nonminimal) S-matrix theories is that of the Dynkin diagram of the *affine* algebra $\hat{\mathscr{G}}$ (the "extended" Dynkin diagram). This symmetry group is [49], in the untwisted cases, the semidirect product of the symmetry of the Dynkin diagram of $\mathscr{G}$ and $Z(\tilde{G})$, the center of the unique simply connected group $\tilde{G}$ whose Lie algebra is $\mathscr{G}$. Note in particular that the decomposition [49] of the dihedral symmetry group D_4 – common to all the $D_n^{(1)}$ ($n \geqslant 5$) Dynkin diagrams – into different semidirect products, depending on the parity of n, is in accordance with the symmetries of the corresponding S-matrices discussed in subsect. 3.3. On the other hand, the same symmetry shows up in the properties of the representations of the related ordinary Lie algebra: The symmetry of the ordinary Dynkin diagram is mirrored by the conjugacy properties of the representations (D_4 is the only case where the Z_2 conjugation is "elevated" to S_3), and the congruency class of a representation [50] corresponds to its $Z(\tilde{G})$ charge. Apart from the fact that the Toda theories which are presumably relevant here are related to affine algebras, the only indication that supports the affine algebra connection is that for the models of subsect. 3.2 the spins of the IMs coincide with the exponents of the twisted affine Lie algebras $A_{2n}^{(2)}$, which are different from those of any ordinary Lie algebra. However, these models are peculiar in many

* The incidence matrix of a Lie algebra is $2 - C$, where C is its Cartan matrix. The Perron–Frobenius vector of a matrix with nonnegative entries is the unique eigenvector all of whose components can be chosen to be positive. It corresponds to a real, nondegenerate eigenvalue, which is not smaller than the magnitude of any other eigenvalue [47].

other respects, and therefore this indication is not absolutely compelling. It is still an open question, precisely which structure should appear – and hopefully be understood – in the classification of diagonal S-matrix theories. For convenience, we will continue to refer to the S-matrix theories under consideration as related to affine Lie algebras.

We mentioned that the eigenvector of the largest eigenvalue of the incidence matrix of $\mathscr{G}$ gives the masses of the $\hat{\mathscr{G}}$-related S-matrix theory. In general, the eigenvalues of the incidence matrix of a simple Lie algebra $\mathscr{G}$ are of the form $2\cos(\pi s/h_{\mathscr{G}})$, where s runs over the exponents of $\mathscr{G}$ [51]. We noticed that the eigenvector corresponding to $2\cos(\pi s/h_{\mathscr{G}})$ is the vector $\boldsymbol{\gamma}_s = (\gamma_s^a)$ assembled out of the coefficients γ_s^a appearing in eq. (16), where a runs over all the particles in the model. In other words, eq. (17) is satisfied for s any exponent (mod $2h_{\mathscr{G}}$) of $\mathscr{G}$ and all a, b and c whose three-point coupling $g_{ab}^c \neq 0$ in the $\hat{\mathscr{G}}$-related S-matrix theory, if we use for $\gamma_s^a = \gamma_{2h_{\mathscr{G}}-s}^a$ the ath component of the eigenvector of $I_{\mathscr{G}}$ with eigenvalue $2\cos(\pi s/h_{\mathscr{G}})$. This is also true for $A_{2n}^{(2)}$, if one uses its "generalized incidence matrix" defined above, provided that $h_{\mathscr{G}}$ is taken to be $2n+1$ and s runs over the first n exponents $1,3,5,\ldots,2n-1$ of $A_{2n}^{(2)}$. Note that for D_n, n even, there are two IMs of spin $s=n-1$, corresponding to the two eigenvectors of eigenvalue 0. One of these two IMs, and the unique IM of spin $n-1$ in the $D_n^{(1)}$ models with n odd, act nontrivially only on the two fundamental particles f and f′, with $\gamma_{n-1}^{\mathrm{f}} = -\gamma_{n-1}^{\mathrm{f}'}$. The sole "purpose" of this IM seems to be to distinguish the two fundamental particles, and thereby make the model reflectionless. We see that in any of the Lie algebra-related S-matrix theories of this section with r particles, there are r linearly independent r-component vectors $\boldsymbol{\gamma}_s$. We therefore conclude that we can distinguish all particles by their charges, as claimed in sect. 2.

Ignoring for the moment the case of $A_{2n}^{(2)}$, the fact that the masses are just some eigenvector of the incidence matrix of $\mathscr{G}$ allows us to uniquely associate the masses to the nodes of the Dynkin diagram of $\mathscr{G}$ (this was also noted in ref. [20], and for E_8 in ref. [16]). Our labeling of the particles in the previous subsections has been chosen accordingly. For A_n our particle labeling corresponds to the standard labeling of notes on the Dynkin diagrams. The same is true for D_n, with f_1, f_2 $(f, \bar{f})$ for n even (odd) corresponding to the nodes usually labeled "$n-1$" and "n" (the spinor representations) of the Dynkin diagram of D_n. For E_6, E_7 and E_8 our labeling particles corresponds to the somewhat unusual labeling of roots indicated in fig. 1.

Recall that the fundamental or primitive [52] representations of a Lie algebra $\mathscr{G}$ (these are the representations whose highest weights are fundamental weights) can also be associated to nodes of the Dynkin diagram of $\mathscr{G}$. Interestingly, if a node corresponds to a primitive representation of greater dimension than another, then (excluding *the* fundamental particles in the $D_n^{(1)}$ models, in general) the particle associated to the first node is heavier than that associated to the second node (cf. ref. [16], again for E_8). Now the reader will not be surprised anymore to learn that

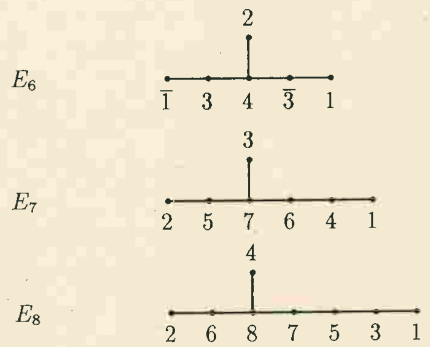

Fig. 1. Our root (particle) ordering for E_6, E_7 and E_8.

the fundamental representation(s)* of the Lie algebras A_n, D_n, E_6, E_7 and E_8 exactly correspond to *the* fundamental particle(s) (in the sense of sect. 2) of the related S-matrix theory.

One might wonder whether the massive fusion rules (see sect. 2) of the $\hat{\mathcal{G}}$-related S-matrix theory are also related to the tensor-product fusion rules of the primitive representations of $\mathcal{G}$. Of course, the massive fusion rules cannot be identical to the tensor-product fusions, because the massive fusions neither have an "identity" nor do they describe particles corresponding to any of the other nonprimitive representations, appearing on the right-hand side of the tensor product of two primitive representations. But one could ask whether the two types of fusion rules coincide if we restrict the tensor-product fusions to primitive representations only on their right-hand side. The massive fusion rules of the $A_n^{(1)}$-related S-matrix theories then in fact equal the restricted tensor-product fusions of A_n. However, for D_n, ($n \geqslant 5$), E_6, E_7 and E_8 the two types of fusion rules do *not* agree: For these Lie algebras the adjoint representation is one of the primitive representations, namely that corresponding to node (particle) number 2 for D_n, E_6 and E_7, and number 1 for E_8. The tensor product of any representation with its conjugate representation always contains the adjoint representation, but the corresponding statement for the massive fusion rules is not true.

The massive fusion rules exhibit a rather peculiar structure. On the one hand they are not associative and have no "identity", which makes them look rather "structureless", but on the other hand they satisfy the following noteworthy regularity (which was also noticed in ref. [20] for ATFTs, by determining the nonzero three-particle couplings in their lagrangians): For every particle c in the

* The fundamental representation(s) of a Lie algebra are the one or two primitive representations, which, by taking tensor products among them, generate all representations of the Lie algebra.

$\hat{\mathscr{G}}$-related minimal or nonminimal theory

$$\sum_{a,b} M^c_{ab} = h_{\mathscr{G}} - 2, \tag{44}$$

where the double sum is taken over all the particles a, b in the model, and in the $\hat{\mathscr{G}} = A^{(2)}_{2n}$ case $h_{\mathscr{G}}$ must again be read as $\frac{1}{2}h_{\hat{\mathscr{G}}} = 2n+1$, as everywhere else in the remainder of this section. Note that for this formula to hold, we must count all odd-order poles as giving rise to bound states.

Finally, we comment on the Z-factors. Assume that the Z-factors of a theory depend continuously on some parameters, and that the full nonminimal S-matrix becomes identically $\mathbb{1}$ as one of the parameters, say b, approaches a certain value (which we can choose to be 0). In other words, the Z-factors allow for a trivial weak coupling limit. Extending an argument in ref. [45], it is then easy to see that for the theories described earlier, the bootstrap equation applied to the fundamental particle(s) uniquely determines the Z-factors to be the ones presented above. Note that these Z-factors depend only on the parameter b. Since none of the Z-factors of the two-particle amplitudes should have poles in the physical strip, b must be restricted to lie in the region $[0, 2/h_{\mathscr{G}}]$ for the $\hat{\mathscr{G}}$-related S-matrix theory. The Z-factors remain unchanged [5,16,22] under the substitution $b \to (2/h_{\mathscr{G}}) - b$. The conjectured [5,16,21,45] explicit dependence of b on the ATFT coupling β (that agrees with low-order perturbative calculations) reads

$$b(\beta) = \frac{2}{h_{\mathscr{G}}} \frac{\beta^2}{1+\beta^2}, \tag{45}$$

in a convenient normalization [22] of β. Assuming it is correct, the Z-factors, and hence the full nonminimal S-matrices, are invariant under the substitution $\beta \to 1/\beta$ which connects the strong and the weak coupling regimes of the quantum affine Toda theories (cf. ref. [22]).

4. The thermodynamic Bethe ansatz and the finite-size scaling coefficient

Consider a critical (classical) statistical system on a torus of perpendicular cycles of length R and L. In the limit of an infinitely long cylinder $L \to \infty$, the free energy per unit length F – equal to the ground-state energy E_0 of the corresponding two-dimensional CFT – is given by [53,54]

$$F = f_0 R - \frac{\pi \tilde{c}}{6R} + \mathrm{o}(\tfrac{1}{R}). \tag{46}$$

Here f_0 is a nonuniversal bulk term, the free energy per unit area in the infinite plane; the second universal term is proportional to what we will call the *finite-size scaling coefficient* $\tilde{c} = c - 12d_0$, where c and d_0 are, respectively, the central charge and the lowest scaling dimension of the CFT to which the critical system renormalizes. For a unitary CFT (on the torus) the lowest scaling dimension is that of the vacuum, $d_0 = 0$, and so $\tilde{c} = c$.

As was demonstrated by Al.B. Zamolodchikov [27], one can calculate the finite-size scaling coefficient $\tilde{c}$ of the UV limit of a purely elastic scattering theory directly from its S-matrix, using the so-called *thermodynamic Bethe ansatz*. This method is, roughly speaking, the usual Bethe ansatz "in reverse". In the usual Bethe ansatz one starts with a hamiltonian formulation of the theory in question, and uses an ansatz for the wave functions to exactly diagonalize the hamiltonian, obtaining the spectrum and sometimes even the S-matrix of the theory (for a review of the Bethe ansatz see ref. [55]). The success of the Bethe ansatz for a given theory is related to the existence of infinitely many IMs. In the thermodynamic Bethe ansatz employed here, this logic is reversed, to accomplish a more limited goal: One uses the *known* spectrum and scattering matrix elements of an integrable QFT to calculate the thermodynamics (equivalently, finite-size effects) of the theory, by assuming that its finite temperature states are described by Bethe ansatz wave functions. We will not try to justify all of the assumptions involved in the thermodynamic Bethe ansatz. However, in order to make this paper relatively self-contained, we would like to outline the steps leading to the main result, eq. (67), for the finite-size scaling coefficient $\tilde{c}$.

The basic idea is as follows. We want to calculate the ground-state energy E_0 of the relativistic QFT described by the purely elastic scattering theory, living on an infinitely long cylinder of circumference R. In standard QFT conventions, where the bulk term vanishes, this ground-state energy has to be of the form

$$E_0(R) = -\frac{\pi \tilde{c}(r)}{6R}, \tag{47}$$

where $\tilde{c}$ (by dimensional arguments) is a function of $r = R/R_c$ alone; $R_c = 1/m_1$ being the largest correlation length, corresponding to the smallest mass m_1 in the theory. We will call $\tilde{c}(r)$ the *finite-size scaling function* of the theory. $\tilde{c}(0)$ is of course the finite-size scaling coefficient $\tilde{c}$ of the CFT describing the UV limit of the massive scattering theory. To calculate $E_0(R)$ using the thermodynamic Bethe ansatz, consider the partition function $Z(R, L)$ of the QFT (in its euclidean version) on a torus of perpendicular cycles of length R and L. There are two natural ways to pass to a hamiltonian formulation of the theory. We can choose time in the L direction. Then, as $L \to \infty$, we have $\ln Z(R, L) \simeq -LE_0(R)$. Alternatively, we can choose time in the R direction. Now, in the $L \to \infty$ limit $Z(R, L)$ is the partition function of the QFT based on an infinite one-dimensional space and periodic in time (period R), or, equivalently, at finite temperature $1/R$ (we use units where $\hbar$, c and Boltzmann's constant are equal to 1). As $L \to \infty$ we now have $\ln Z(R, L) \simeq -LRf(R)$, where $f(R)$ is the free energy per unit length of our QFT at finite temperature $1/R$. It follows that $E_0(R) = Rf(R)$. We then assume that the finite temperature states of the QFT are described by Bethe ansatz wave functions (all the information we need to know about these wave functions is contained in the scattering matrix elements of the theory, as will be explained below). This allows us to obtain the energy levels of the system, and thermodynamics then gives the free energy $f(R)$. The relation $E_0(R) = Rf(R)$ and the $R \to 0$ limit of eq. (47) finally yield the finite-size scaling coefficient $\tilde{c}$.

In more detail, consider the (finite temperature) QFT on a space of length L with periodic boundary conditions. Let n denote the number of different particle species in the theory. Consider N particles labelled $i_1, \ldots, i_N$, N_a of which are of species a, at positions $x_{i_1}, \ldots, x_{i_N}$. The space of all possible configurations of these particles decomposes into $N!$ regions, corresponding to the $N!$ orderings of the particles. Each region contains a so-called "free region", in which all particles are far away from each other, i.e. $|x_{i_k} - x_{i_l}| \gg R_c$ for all k and l, and behave like free particles with well-defined momenta p_{i_k}. So, in the free regions the wave function is just proportional to that of free particles. Besides this fact, all we have to know is the following: The wave function in the free region where $x_{i_j} \ll x_{i_{j+1}}$, is obtained from the one in the neighboring free region where $x_{i_j} \gg x_{i_{j+1}}$ by multiplication by $S_{i_j i_{j+1}}(\theta_{i_j} - \theta_{i_{j+1}})$. Here θ_{i_j} and $\theta_{i_{j+1}}$ are the rapidities in the free regions. The requirement that the wave function be periodic (period L) in its N arguments, leads to

$$e^{iLm_i \sinh\theta_i} \prod_{j:\, j\neq i} S_{ij}(\theta_i - \theta_j) = 1, \qquad \text{for } i = 1, 2, \ldots, N. \tag{48}$$

From now on we drop the double index notation for particles, as we already did in the last equation, and assume that the θ_i are ordered, $\theta_i \leqslant \theta_{i+1}$ for all $i = 1, 2, \ldots, N-1$. Introducing the phase shifts $\delta_{ij}(\theta_i - \theta_j) = -i \ln S_{ij}(\theta_i - \theta_j)$, the logarithm of eq. (48) can be written as

$$Lm_i \sinh\theta_i + \sum_{j:\, j\neq i} \delta_{ij}(\theta_i - \theta_j) = 2\pi n_i \qquad \text{for } i = 1, 2, \ldots, N, \tag{49}$$

with the n_i some integers. So we see that periodic boundary conditions, rather trivial for a free theory, lead to a set of N coupled transcendental equations for the rapidities in an interacting theory, the *Bethe ansatz equations*. We stress that these equations involve the physical particles in the spectrum of the theory, not the pseudo-particles of the usual Bethe ansatz method [55], and consequently all the rapidities θ_i are real.

The thermodynamics of the system will be dominated by certain distributions of the rapidities θ_i and the n_i, with the distributions of these quantities constrained by eq. (49). Before we can determine these equilibrium distributions (in the thermodynamic limit), we have to take into account the constraints on the allowed rapidities arising from the identity of particles.

The N-particle wave function must be symmetric (antisymmetric) under the interchange of identical bosons (fermions) of the same rapidity. Let particle a be a boson or a fermion. From the unitarity condition (4) we see that $S_{aa}^2(0) = 1$. So there are two possibilities:

(i) $S_{aa}(0) = -1$: This is incompatible with the symmetry of the wave function under interchange of identical bosons a with the same rapidity. Therefore, if $S_{aa}(0) = -1$ two bosons of species a are not allowed to have the same rapidity. We will refer to such particles as being of *fermionic type*. If a is a fermion there is no restriction and we will say that it is of *bosonic type*.

(ii) $S_{aa}(0) = 1$: Here the situation is reversed. Identical fermions are not allowed to have the same rapidity, i.e. are of fermionic type, and bosons are of bosonic type.

The above discussion can be summarized by introducing the *type* of a particle. If $(-1)^{F_a} = \pm 1$ indicates if a is a boson or a fermion, respectively, the type of a is defined by $t_a = -(-1)^{F_a} S_{aa}(0)$. Then $t_a = \pm 1$ corresponds to a particle of fermionic or bosonic type, respectively. In theories with one space dimension the statistics of a particle is physically irrelevant – at least if the particle is a fermion or a boson – because it becomes inextricably mixed up with the interaction: As we have just seen, for instance, an exclusion principle holds in a system of bosons with zero-rapidity scattering amplitude -1. In fact, this is the significance of the type of a particle – it tells us whether an exclusion principle holds. In this sense, particles of fermionic and bosonic type in one space dimension are the analogues of fermions and bosons, respectively, in higher dimensions. We should remark that in most of the "restricted sine-Gordon models" mentioned earlier, some of the particles seem to obey "exotic" statistics [24, 25]. The thermodynamic Bethe ansatz remains to be generalized to such theories.

Let us also mention that it is not clear if particles of bosonic type can actually exist in any consistent interacting theory. Within the context of the "true" Bethe ansatz for the nonlinear Schrödinger model, it has been shown explicitly that identical rapidities do not occur; this also appears to be necessary to construct the physical vacuum in this and other theories [57]. We will see in sect. 5 that all particles in the models described in sect. 3 seem to be of fermionic type. For the sine-Gordon model this can of course be explicitly checked, because there we not only know the S-matrix elements, but also which particles are fermions and which are bosons. One would certainly like to prove in general that an exclusion principle holds for all particles in any interacting theory in $1+1$ dimensions – or give a counterexample.

Eq. (49) can be analyzed in the thermodynamic limit where both L and all N_a become very large, and the N_a/L are finite. We can then introduce the density $\rho_r^{(a)}(\theta)$ as the number of particles of species a with rapidities between $\theta - \frac{1}{2}\Delta\theta$ and $\theta + \frac{1}{2}\Delta\theta$ divided by $L\,\Delta\theta$. We are assuming that it is possible to choose the intervals $\Delta\theta$ ($\Delta\theta$ can depend on θ) large enough to have an appreciable number of rapidity levels in them, but small enough so that the $\rho_r^{(a)}(\theta)$ vary only on a scale larger than several $\Delta\theta$. Let us also introduce for each fixed $a = 1, \ldots, n$ the subsets $n_i^{(a)}$ of the set of all the n_i in eq. (49), where i is now running only over the particles of species a. Let $\theta_i^{(a)}$ be the rapidity values corresponding to these $n_i^{(a)}$. Eq. (49) can then be written as

$$m_a \sinh\theta_i^{(a)} + \sum_{b=1}^{n} \int_{-\infty}^{+\infty} \mathrm{d}\theta'\, \delta_{ab}\big(\theta_i^{(a)} - \theta'\big)\rho_r^{(b)}(\theta') = \frac{2\pi n_i^{(a)}}{L}\,. \tag{50}$$

Consider the functions $J^{(a)}(\theta)$ defined by

$$J^{(a)}(\theta) = \frac{m_a}{2\pi}\sinh\theta + \sum_{b=1}^{n} \left(\delta_{ab} * \rho_r^{(b)}\right)(\theta)\,, \tag{51}$$

where $*$ denotes the convolution

$$(\delta * \rho)(\theta) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\theta'}{2\pi}\delta(\theta - \theta')\rho(\theta')\,. \tag{52}$$

Once we know more about the densities $\rho_r^{(a)}$ we will see that these functions are monotonically increasing functions of θ. Assume this to be true for the moment. Then the sequence of $n_i^{(a)}$ is monotonically increasing with i. We see that if $J^{(a)}(\theta) = n_i^{(a)}/L$, then $\theta = \theta_i^{(a)}$. Such a θ is called a *root* of species a. Note that the density of roots of species a is $\rho_r^{(a)}$. If the increasing sequence of n_i skips some integers there will be values of θ not among the $\theta_i^{(a)}$ such that $LJ^{(a)}(\theta)$ equals these skipped integers. Such values of θ will be called *holes* of species a. (For bosonic type particles "an integer is skipped" even if $n_{i+1}^{(a)} - n_i^{(a)} = 1$, see below.)

For particles of fermionic type the $n_i^{(a)}$ must form a strictly increasing sequence of integers because these particles are not allowed to have the same rapidity. So we see that $n_{i+1}^{(a)} - n_i^{(a)} = 1 + n_{h,i}^{(a)} \geqslant 1$, where $n_{h,i}^{(a)}$ is the number of holes of species a between the rapidities $\theta_i^{(a)}$ and $\theta_{i+1}^{(a)}$. We can therefore define a density of states (roots *and* holes) for the particles of species a, $\rho^{(a)} = \rho_{\mathrm{r}}^{(a)} + \rho_{\mathrm{h}}^{(a)}$, by

$$\rho^{(a)}(\theta) \equiv \frac{\mathrm{d}}{\mathrm{d}\theta} J^{(a)}(\theta) = \frac{m_a}{2\pi}\cosh\theta + \sum_{b=1}^{n} \left(\varphi_{ab} * \rho_{\mathrm{r}}^{(b)}\right)(\theta)\,, \tag{53}$$

where $\varphi_{ab}(\theta)$ is given by

$$\varphi_{ab}(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}\delta_{ab}(\theta) = -i\frac{\mathrm{d}}{\mathrm{d}\theta}\ln S_{ab}(\theta)\,. \tag{54}$$

The explicit form of the $\varphi_{ab}(\theta)$ in purely elastic scattering theories will be given at the end of this section. Here we just note that the unitarity of the S-matrix, eq. (4), implies that $\varphi_{ab}(-\theta) = \varphi_{ab}(\theta)$.

On the other hand, if the particles a are of bosonic type they are allowed to have the same rapidity values, and in this case $n_{i+1}^{(a)} - n_i^{(a)} = n_{\mathrm{h},i}^{(a)} \geqslant 0$. Here the density of states $\rho^{(a)}(\theta)$, defined by the same equation as above, equals $\rho_{\mathrm{h}}^{(a)}(\theta)$.

Recall our assumption that the rapidity axis can be divided into intervals of size $\Delta\theta$ over which the densities do not vary appreciably, and the number of roots $L\rho_{\mathrm{r}}^{(a)}(\theta)\,\Delta\theta$ and the number of holes $L\rho_{\mathrm{h}}^{(a)}(\theta)\,\Delta\theta$ in each of these intervals is large. Hence there are

$$\frac{\left[L\left(\rho_{\mathrm{r}}^{(a)} + \rho_{\mathrm{h}}^{(a)}\right)(\theta)\,\Delta\theta\right]!}{\left[L\rho_{\mathrm{r}}^{(a)}(\theta)\,\Delta\theta\right]!\left[L\rho_{\mathrm{h}}^{(a)}(\theta)\,\Delta\theta\right]!} \tag{55}$$

ways to distribute the roots (particles) and holes in the interval $\Delta\theta$ giving the same "macroscopic" densities $\rho_{\mathrm{r}}^{(a)}$ and $\rho_{\mathrm{h}}^{(a)}$. The logarithm of this expression gives the contribution of $\Delta\theta$ to the entropy★. The total entropy per unit length is thus given by

$$\begin{aligned} s[\rho_{\mathrm{h}}, \rho_{\mathrm{r}}] &= \sum_{a=1}^{n} s_a\left[\rho_{\mathrm{h}}^{(a)}, \rho_{\mathrm{r}}^{(a)}\right] \\ &= \sum_{a=1}^{n}\int_{-\infty}^{+\infty}\mathrm{d}\theta\left[\left(\rho_{\mathrm{r}}^{(a)} + \rho_{\mathrm{h}}^{(a)}\right)\ln\left(\rho_{\mathrm{r}}^{(a)} + \rho_{\mathrm{h}}^{(a)}\right) - \rho_{\mathrm{r}}^{(a)}\ln\rho_{\mathrm{r}}^{(a)} - \rho_{\mathrm{h}}^{(a)}\ln\rho_{\mathrm{h}}^{(a)}\right]. \end{aligned} \tag{56}$$

Since the relation between $\rho^{(a)}$ and $\rho_{\mathrm{r}}^{(a)}$, eq. (53), holds for both types of particles, it is convenient to consider s_a as a functional of $\rho^{(a)}$ and $\rho_{\mathrm{r}}^{(a)}$. The functional form

★ The following variational method to calculate the thermodynamics of a system solvable by the Bethe ansatz was introduced by Yang and Yang [56].

of $s_a[\rho^{(a)}, \rho_r^{(a)}]$ is then different for particles of fermionic and bosonic type and is obtained by substituting $\rho_h^{(a)} = \rho^{(a)} - \rho_r^{(a)}$ and $\rho_h^a = \rho^{(a)}$, respectively, in these cases.

Using the thermodynamic relation $F = E - TS$ (here $T = 1/R$), the free energy per unit length f is determined by minimizing it as a functional of the $\rho^{(a)}$ and $\rho_r^{(a)}$, these densities being constrained by eq. (53). Upon minimizing

$$Rf[\rho, \rho_r] = Rh[\rho_r] - s[\rho, \rho_r], \tag{57}$$

where

$$h[\rho_r] = \sum_{a=1}^{n} \int_{-\infty}^{+\infty} d\theta \, \rho_r^{(a)}(\theta) \, m_a \cosh\theta \tag{58}$$

is the energy per unit length of a given rapidity distribution, one finds that the extremum condition involves only the ratios $\rho_r^{(a)}(\theta)/\rho^{(a)}(\theta)$. Let us introduce $\epsilon_a(\theta)$ by

$$\frac{\rho_r^{(a)}(\theta)}{\rho^{(a)}(\theta)} = \frac{e^{-\epsilon_a(\theta)}}{1 \pm e^{-\epsilon_a(\theta)}}, \tag{59}$$

and define

$$L_a(\theta) = \pm \ln(1 \pm e^{-\epsilon_a(\theta)}), \tag{60}$$

where here and in the following equations the upper and lower signs refer to the particle a being of fermionic or bosonic type, respectively. Then the extremum condition can be written in the unified form

$$Rm_a \cosh\theta = \epsilon_a(\theta) + \sum_{b=1}^{n} (\varphi_{ab} * L_b)(\theta) \tag{61}$$

for both types of particles. The extremal free energy per unit length $f = f(R)$ is then determined by inserting eqs. (58), (61), and (53) into (57). Doing this, and using $E_0(R) = Rf(R) = -\pi\tilde{c}(r)/6R$, we obtain the following expression for the finite-size scaling function (with $r = R/R_c$):

$$\tilde{c}(r) = \frac{3}{\pi^2} \sum_{a=1}^{n} \int_{-\infty}^{+\infty} d\theta \, L_a(\theta) \hat{m}_a r \cosh\theta. \tag{62}$$

Here the $\epsilon_a(\theta)$ are determined by the nonlinear integral equation (61), and we have normalized the masses such that the smallest mass is 1, i.e. $\hat{m}_a = m_a/m_1 = m_a R_c$.

The $r \to 0$ limit of this expression can be explicitly evaluated, as we will see presently. By taking the derivative of eq. (61) with respect to θ we see that $\epsilon_a(\theta)$ becomes constant in the region $-\ln(2/r) \ll \theta \ll \ln(2/r)$ as $r \to 0$. Let ϵ_a be the value of $\epsilon_a(\theta)$ in this "flat region". (For the nonminimal S-matrices the ϵ_a do not reach a finite limit as $r \to 0$, rather, they diverge like $\ln(m_a r)$ in this limit. Nevertheless, all calculations presented here go through as for the minimal S-matrices where the ϵ_a are finite, see sect. 5.) Then

$$\epsilon_a = \pm \sum_{b=1}^{n} N_{ab} \ln(1 \pm e^{-\epsilon_b}), \tag{63}$$

where we introduced the symmetric matrix N by

$$N_{ab} = -\int_{-\infty}^{+\infty} \frac{d\theta}{2\pi} \varphi_{ab}(\theta) = -\frac{1}{2\pi}(\delta_{ab}(\infty) - \delta_{ab}(-\infty)). \tag{64}$$

In the cases we are considering, there is always a unique real solution to eq. (63) for the ϵ_a (see below). We also see that eq. (61) implies that the $\epsilon_a(\theta)$ are real functions for any r. This proves our previous remark that the $J^{(a)}(\theta)$ of eq. (51) are monotonically increasing functions for the case of fermionic type particles: By definition of $\rho^{(a)}(\theta)$, eq. (53), this is equivalent to showing that $\rho^{(a)}(\theta) > 0$. But since $\rho_r^{(a)}(\theta)$ is manifestly positive, the reality of $\epsilon_a(\theta)$ implies via eq. (59) the positivity of $\rho^{(a)}(\theta)$ for all a. Generally, the interpretation of $\rho^a(\theta)$ as the density of states requires it to be positive in a consistent theory.

Let us return to the evaluation of the $r \to 0$ limit of eq. (62). Since $\epsilon_a(\theta)$ and therefore $L_a(\theta)$ are even in θ, cf. eq. (61), we can replace the lower boundary of the integral in (62) by 0 and multiply it by 2. We then note that the size of the region of integration where $\cosh\theta$ can *not* be approximated by $\frac{1}{2}e^\theta$ is of order 1, and furthermore the integrand there goes to 0 as $r \to 0$, hence this region gives a negligible contribution to the whole integral. Outside this region we can replace $\cosh\theta$ by $\frac{1}{2}e^\theta$, and also replace $\epsilon_a(\theta)$ and $L_a(\theta)$ by $\tilde{\epsilon}_a(\theta)$ and $\tilde{L}_a(\theta) = \pm\ln(1 \pm e^{-\tilde{\epsilon}_a(\theta)})$, respectively, which are determined from

$$\tfrac{1}{2} r\hat{m}_a e^\theta = \tilde{\epsilon}_a(\theta) + \sum_{b=1}^{n} (\varphi_{ab} * \tilde{L}_b)(\theta). \tag{65}$$

Note that the $\tilde{\epsilon}_a(\theta)$ and $\tilde{L}_a(\theta)$ are now constant for all $\theta \ll \ln(2/r)$, and since $\tilde{\epsilon}_a(\theta)$ grows exponentially as $\theta \to \infty$ (cf. eq. (61)), $\tilde{L}_a(\theta)$ decays as a double exponential in this limit. At this point we have

$$\tilde{c}(0) = \frac{6}{\pi^2} \sum_{a=1}^{n} \lim_{r\to 0} \int_0^\infty d\theta\, \tilde{L}_a(\theta) \tfrac{1}{2} r\hat{m}_a e^\theta. \tag{66}$$

We now replace $\frac{1}{2} r\hat{m}_a e^\theta$ in this equation by the derivative of the right-hand side of eq. (65) with respect to θ. To find $\tilde{c} = \tilde{c}(0)$, one then needs only general properties of $\tilde{L}_a(\theta)$ and $\varphi_{ab}(\theta)$, namely that $\tilde{L}_a(\theta)$ is constant for $\theta \ll \ln(2/r)$, it decays rapidly for $\theta \ll \ln(2/r)$, and that $\varphi_{ab}(\theta)$ is an even function of θ which falls off exponentially as one goes outside a region of order 1 around $\theta = 0$. After a few integrations by parts, we arrive at our final result:

$$\tilde{c} = \sum_{a=1}^{n} \tilde{c}_a = \sum_{a=1}^{n} \tilde{c}_{t_a}(\epsilon_a), \tag{67a}$$

where

$$\tilde{c}_{\pm}(\epsilon) = \frac{6}{\pi^2} \times \begin{cases} L\left(\dfrac{1}{1+\mathrm{e}^{\epsilon}}\right) \\ L(\mathrm{e}^{-\epsilon}) \end{cases} = \frac{6}{\pi^2} \int_0^{\infty} \mathrm{d}x \frac{x+\epsilon/2}{\mathrm{e}^{x+\epsilon} \pm 1}. \tag{67b}$$

Here the ϵ_a are determined by eq. (63), t_a is the type of the particle a as defined earlier in this section, and $L(x)$ is Rogers' dilogarithm function [58]

$$L(x) = -\frac{1}{2} \int_0^x \mathrm{d}y \left[\frac{\ln y}{1-y} + \frac{\ln(1-y)}{y} \right]. \tag{68}$$

We see that we can associate a finite-size scaling coefficient $\tilde{c}_a$ to each particle species in the scattering theory, with the total $\tilde{c}$ given by their sum. The individual $\tilde{c}_a$ are obtained by evaluating at ϵ_a either one of two universal functions $\tilde{c}_{\pm}(\epsilon)$, depending on whether a is of fermionic or bosonic type, respectively. Note the following properties of these functions: $\tilde{c}_+(\epsilon)$ is defined for all real ϵ, $\tilde{c}_-(\epsilon)$ for positive ϵ. Both are strictly monotonically decreasing functions, approaching 0 as $\epsilon \to +\infty$; $\tilde{c}_-(0) = 1$ and $\tilde{c}_+(-\infty) = 2\tilde{c}_+(0) = 1$. In eq. (67) we gave two alternative expressions for $\tilde{c}_{\pm}$. The first is useful because known sum rules for the Rogers dilogarithm function can be used to evaluate $\tilde{c}$ in certain models, as we will see in sect. 5. The second form is more suggestive, however. It shows that $\pi\tilde{c}_a/3R$ is the entropy per unit length of a one-dimensional idea quantum gas of massless particles (dispersion relation energy = |momentum|), at temperature $T = 1/R$ and chemical potential $-\epsilon_a/R$. Note that in agreement with earlier remarks in this section, particles of fermionic (bosonic) type are taken to correspond to fermions (bosons) in the calculation of the entropy.

Eq. (63) determining the ϵ_a involves the matrix $N = (N_{ab})$, eq. (64). Since for our diagonal S-matrix theories the $S_{ab}(\theta)$ are products of the building blocks $f_\alpha(\theta)$ of sect. 2, we only have to calculate $\varphi_{ab}(\theta)$ and N_{ab} for the $f_\alpha(\theta)$. If $S_{ab}(\theta) = \Pi_i f_{\alpha_i}(\theta)$, then $\varphi_{ab}(\theta) = \Sigma_i \varphi[f_{\alpha_i}](\theta)$ and $N_{ab} = \Sigma_i N[f_{\alpha_i}]$, in an obvious notation. A simple calculation gives

$$\varphi[f_\alpha](\theta) = -i \frac{\mathrm{d}}{\mathrm{d}\theta} \ln f_\alpha(\theta) = -\frac{\sin \alpha\pi}{\cosh\theta - \cos\alpha\pi},$$

$$N[f_\alpha] = (1 - |\alpha|)\operatorname{sgn}\alpha \qquad \text{for } -1 < \alpha \leqslant 1, \tag{69}$$

where sgn α is the sign of α (we set sgn $0 = 0$). This implies

$$N[F_\alpha] = \operatorname{sgn}\alpha \qquad \text{for } -\tfrac{1}{2} \leqslant \alpha \leqslant \tfrac{1}{2}. \tag{70}$$

We see that each f_α with $0 < \alpha < 1$ gives a positive contribution to N. All the minimal S-matrices of sect. 3 therefore have $N_{ab} > 0$ for all a and b (except for the $A_1^{(1)}$-related minimal model where $N = N_{11} = 0$). It is easy to see that in such a case eq. (63) for the ϵ_a has a unique real solution. Furthermore, in all the S-matrix theories considered in this paper the building blocks f_α occur with rational α. Then N_{ab} is rational for all a and b, implying that the e^{ϵ_a} are algebraic.

5. Calculation of $\tilde{c}$ for purely elastic S-matrices

We now apply the general formalism of sect. 4 to the explicit S-matrix theories discussed in sect. 3, thus finding the finite-size scaling coefficient of the UV limits of each of these theories. For each theory we have to calculate the matrix (N_{ab}), eqs. (64) and (69), find the unique real solution ϵ_a of eq. (63), and finally sum up the contributions $\tilde{c}_a$ of each particle to the total scaling coefficient $\tilde{c}$, eq. (67). The $\tilde{c}_a$ depend on the type, fermionic or bosonic, of each particle (there are no "exotic particles" [cf. sect. 4] in the theories we consider). We know that all particles in the affine Toda field theories are bosons. Since $S_{aa}(0) = -1$ for all particles a in any of the Lie algebra-related models of sect. 3 (with or without Z-factors), this implies that in the S-matrix theories with Z-factors, which presumably describe ATFTs, all particles are of fermionic type. For the minimal S-matrices we do not know a priori if the particles they describe are bosons or fermions. But there are indications that in any intereacting massive QFT in 1 + 1 dimensions all particles must obey an exclusion principle [cf. sect. 4], and, in any case, checking various possibilities, we found that the only ones that lead to any "nice" values for $\tilde{c}$ are those where all particles are of fermionic type. (So that also in the minimal theories all particles are bosons.) Now to the explicit results.

5.1. MINIMAL THEORIES

Our results for the N-matrices can be summarized by the remarkable relation

$$N = I(2 - I)^{-1}. \tag{71}$$

Here I is the incidence matrix of the ordinary (non-affine) algebra in question, with the nodes of the corresponding Dynkin diagram labeled by the particles of the S-matrix theory, as discussed in sect. 3. In the exceptional $A_{2n}^{(2)}$-related cases, I is the "generalized incidence matrix" introduced in sect. 3. For the infinite families $A_n^{(1)}$, $A_{2n}^{(2)}$ and $D_n^{(1)}$ the results for the N-matrices and the ϵ_a below are generalizations of those found numerically in a finite number of cases that was large enough sh-Gordon ($A_1^{(1)}$) $e^{\epsilon_1} = \dfrac{\sin\frac{3\pi}{4}}{\sin \pi/4} = 1 = 1/x_i$; $c = \frac{\pi^2}{6}\mathcal{L}\left(\frac{1}{2}\right) = 1$

to establish a general rule (it is then only a matter of tedious algebra to prove these results directly). The resulting values for $\tilde{c}$ follow from some remarkable sum rules for the Rogers dilogarithm [58]. In the three $E_n^{(1)}$-related cases we evaluated $\tilde{c}$ numerically; apparently, the corresponding sum rules are not known in the literature. Following are our results for the ϵ_a and the $\tilde{c}$. We state the values of the e^{ϵ_a}, which are, as we already mentioned, algebraic numbers. The values of $\tilde{c}$ are, in all cases, the finite-size scaling coefficients of the relevant CFTs, discussed in sect. 3.

5.1.1. $A_n^{(1)}$. Here the real solution of eq. (63) is

$$e^{\epsilon_a} = \sum_{k=1}^{a}\left[\sin\frac{(2k+1)\pi}{n+3}\right]\Big/\sin\left[\left(\frac{\pi}{n+3}\right)\right] = \sin\frac{a\pi}{n+3}\sin\left(\frac{(a+2)\pi}{n+3}\right)\Big/\sin^2\left(\frac{\pi}{n+3}\right)$$

$$\text{for } a = 1,2,\ldots,n, \quad (72)$$

and the above mentioned sum rules [58] lead to

$$\tilde{c} = \frac{2n}{n+3}, \quad (73)$$

in agreement with the central charge $c = 2n/(n+3)$ of the unitary CFT describing Z_{n+1} parafermions (see sect. 3). For $n=2$, eq. (72) reduces to the result of ref. [27], where $\tilde{c} = 2\tilde{c}_1 = \frac{4}{5}$.

5.1.2. $A_{2n}^{(2)}$. As noted by Al.B. Zamolodchikov [27] in the $n=1$ case, the factorization (25) of $A_{2n}^{(2)}$ scattering amplitudes into $A_{2n}^{(1)}$ amplitudes implies

$$e^{\epsilon_a}\left(A_{2n}^{(2)}\right) = e^{\epsilon_a}\left(A_{2n}^{(1)}\right) = \sin\left(\frac{a\pi}{2n+3}\right)\sin\left(\frac{(a+2)\pi}{2n+3}\right)\Big/\sin^2\left(\frac{\pi}{2n+3}\right)$$

$$\text{for } a = 1,2,\ldots,n. \quad (74)$$

Consequently,

$$\tilde{c}_a\left(A_{2n}^{(2)}\right) = \tilde{c}_a\left(A_{2n}^{(1)}\right) = \tilde{c}_{2n+1-a}\left(A_{2n}^{(1)}\right) \qquad \text{for } a = 1,2,\ldots,n, \quad (75)$$

and $\tilde{c}(A_{2n}^{(2)}) = \frac{1}{2}\tilde{c}(A_{2n}^{(1)}) = 2n/(2n+3)$. Indeed, these are the values of $\tilde{c}$ obtained for the nonunitary minimal models of central charge $c = -2n(6n+5)/(2n+3)$ and lowest scaling dimension $d_0 = -n(n+1)/(2n+3)$.

5.1.3. $D_n^{(1)}$. Here

$$e^{\epsilon_a} = \sum_{k=1}^{a}(2k+1) = a(a+2) \qquad \text{for } a = 1,2,\ldots,n-2,$$

$$e^{\epsilon_f} = n-1 \qquad \text{where } f = f_1, f_2\ (f,\bar{f}) \text{ for } n \text{ even (odd)}. \quad (76)$$

For $n=4$, for instance, we find

$$\tilde{c}_1 = \tilde{c}_{f_1} = \tilde{c}_{f_2} = 0.283937615\ldots, \qquad \tilde{c}_2 = 0.148187153\ldots, \tag{77}$$

which add up to give $\tilde{c}=1$, as for all $n \geqslant 2$. We remark that in general, unlike in the symmetry-enhanced $n=4$ case, the mass degeneracy $m_k = m_f$ in the $\mathrm{D}^{(1)}_{3k+1}$ S-matrix theory ($k>1$ a positive integer) does not lead to $\tilde{c}_k = \tilde{c}_f$.

Note that the same calculations as above show that $\tilde{c}=1$ for the sine-Gordon model at its reflectionless points – as expected. The calculation is identical, because (i) the different signs in some of the sine-Gordon amplitudes (see discussion at the end of subsect. 3.3) do not affect the N-matrix, and (ii) although the soliton and antisoliton of the sine-Gordon model are fermions, their zero-rapidity amplitudes now satisfy $S_{ss}(0) = S_{\bar{s}\bar{s}}(0) = +1$, so that again all particles are of fermionic type.

5.1.4. $\mathrm{E}^{(1)}_6$. In this case we find

$$\mathrm{e}^{\epsilon_1} = \mathrm{e}^{\epsilon_{\bar{1}}} = \frac{\sin\frac{2}{7}\pi}{\sin\frac{1}{7}\pi} + \frac{\sin\frac{3}{7}\pi}{\sin\frac{1}{7}\pi}, \qquad \mathrm{e}^{\epsilon_2} = 1 + 2\frac{\sin\frac{2}{7}\pi}{\sin\frac{1}{7}\pi} + \frac{\sin\frac{3}{7}\pi}{\sin\frac{1}{7}\pi},$$

$$\mathrm{e}^{\epsilon_3} = \mathrm{e}^{\epsilon_{\bar{3}}} = 1 + 3\frac{\sin\frac{2}{7}\pi}{\sin\frac{1}{7}\pi} + 4\frac{\sin\frac{3}{7}\pi}{\sin\frac{1}{7}\pi}, \qquad \mathrm{e}^{\epsilon_4} = 5 + 9\frac{\sin\frac{2}{7}\pi}{\sin\frac{1}{7}\pi} + 11\frac{\sin\frac{3}{7}\pi}{\sin\frac{1}{7}\pi}, \tag{78}$$

and

$$\tilde{c}_1 = \tilde{c}_{\bar{1}} = 0.235595207\ldots, \qquad \tilde{c}_2 = 0.165390903\ldots,$$
$$\tilde{c}_3 = \tilde{c}_{\bar{3}} = 0.091171917\ldots, \qquad \tilde{c}_4 = 0.038217704\ldots. \tag{79}$$

This gives $\tilde{c} = \frac{6}{7}$.

5.1.5. $\mathrm{E}^{(1)}_7$. Here the results are

$$\mathrm{e}^{\epsilon_1} = 2+\sqrt{5}, \qquad \mathrm{e}^{\epsilon_2} = \tfrac{1}{2}(5+3\sqrt{5}), \qquad \mathrm{e}^{\epsilon_3} = 6+3\sqrt{5},$$
$$\mathrm{e}^{\epsilon_4} = 8+4\sqrt{5}, \qquad \mathrm{e}^{\epsilon_5} = \tfrac{1}{2}(33+15\sqrt{5}), \qquad \mathrm{e}^{\epsilon_6} = 27+12\sqrt{5},$$
$$\mathrm{e}^{\epsilon_7} = 80+36\sqrt{5}. \tag{80}$$

Further, we find

$$\tilde{c}_1 = 0.228828328\ldots, \qquad \tilde{c}_2 = 0.184429673\ldots, \qquad \tilde{c}_3 = 0.105461151\ldots,$$
$$\tilde{c}_4 = 0.084686687\ldots, \qquad \tilde{c}_5 = 0.049684107\ldots, \qquad \tilde{c}_6 = 0.033540408\ldots,$$
$$\tilde{c}_7 = 0.013369644\ldots, \tag{81}$$

and it follows that $\tilde{c} = \frac{7}{10}$.

5.1.6. $E_8^{(1)}$. Finally, in this case

$$\mathrm{e}^{\epsilon_1} = 2 + 2\sqrt{2}\,, \qquad \mathrm{e}^{\epsilon_2} = 5 + 4\sqrt{2}\,, \qquad \mathrm{e}^{\epsilon_3} = 11 + 8\sqrt{2}\,,$$
$$\mathrm{e}^{\epsilon_4} = 16 + 12\sqrt{2}\,, \qquad \mathrm{e}^{\epsilon_5} = 42 + 30\sqrt{2}\,, \qquad \mathrm{e}^{\epsilon_6} = 56 + 40\sqrt{2}\,,$$
$$\mathrm{e}^{\epsilon_7} = 152 + 108\sqrt{2}\,, \qquad \mathrm{e}^{\epsilon_8} = 543 + 384\sqrt{2}\,. \tag{82}$$

The "partial" finite-size scaling coefficients are

$$\tilde{c}_1 = 0.210009676\ldots, \qquad \tilde{c}_2 = 0.120269807\ldots, \qquad \tilde{c}_3 = 0.068324712\ldots,$$
$$\tilde{c}_4 = 0.050048381\ldots, \qquad \tilde{c}_5 = 0.023056286\ldots, \qquad \tilde{c}_6 = 0.018087052\ldots,$$
$$\tilde{c}_7 = 0.007688924\ldots, \qquad \tilde{c}_8 = 0.002515159\ldots, \tag{83}$$

and their sum is $\tilde{c} = \frac{1}{2}$.

We note that, with the exception of the fundamental particles in the $D_n^{(1)}$ theories, the ϵ_a in each theory increase with the masses m_a of the particles; whence $\tilde{c}_a$ is smaller the heavier the particle a is (in a given theory), with the abovementioned exceptions.

5.2. NONMINIMAL THEORIES

Next we turn to the S-matrices conjectured for the ATFTs, with the Z-factors of sect. 3 included. In this case the structure of the Z-factors leads to the simple b-independent result (as long as $0 < b < 2/h_{\mathscr{G}}$)

$$N = -\mathbb{1} \tag{84}$$

in all models, where $\mathbb{1}$ is the identity matrix. Since all particles are of fermionic type (if any particle were of bosonic type eq. (63) would not even have a real solution for $N = -\mathbb{1}$), this implies $\epsilon_a = -\infty$, hence $\tilde{c}_a = 1$, for all particles a in all the theories. It follows that $\tilde{c} = \mathrm{rank}(\mathscr{G})$ for the $\hat{\mathscr{G}}$-related nonminimal S-matrix theories. This is exactly the behavior expected of real-coupling affine Toda field theories (at least in the simply laced cases), since in the UV, where all masses can be neglected, they become free. This follows from the fact that the interaction term is multiplied by $(m_0/\beta)^2$, where m_0 is the mass scale, and the fact that β does not renormalize.

To conclude this section, we remark that there is an ambiguity in the N-matrix and type of free particles. Take for instance the weak coupling limit of an ATFT. We have just seen that for an arbitrarily small but still strictly positive b, $N = -\mathbb{1}$

and $\epsilon_a = -\infty$ for all particles, which are of fermionic type. On the other hand, for the free theory the S-matrix is trivial, $\mathscr{S} \equiv \mathbb{1}$, hence all particles are of bosonic type and $N = 0$, consequently $\epsilon_a = 0$ for all particles. This apparently different result arises because $S_{aa}(0) = -1$ for all $b(\beta) > 0$; the S-matrix with Z-factors does not approach the free S-matrix $\mathscr{S} \equiv \mathbb{1}$ uniformly in the weak coupling limit. There is of course no contradiction in the final result; the contribution of each particle to $\tilde{c}$ is the same in both interpretations, since $\tilde{c}_+(-\infty) = \tilde{c}_-(0) = 1$.

6. Conclusions and discussion

We presented a method for calculating the finite-size scaling coefficient $\tilde{c}$ of an arbitrary diagonal scattering theory. Applying this method, we calculated $\tilde{c}$ for the minimal S-matrix theories of sect. 3, that were proposed to be related to massive perturbations of certain rational CFTs. In each case we found that $\tilde{c}$ equals the value of $c - 12d_0$ of the corresponding CFT. Previous work provided strong evidence that the minimal S-matrices are the minimal part of the full S-matrices of the massive QFTs obtained by perturbing these CFTs. Our results leave hardly any doubt that the minimal S-matrices are in fact the *complete* S-matrices of these theories. If one amends the minimal S-matrices with the unique Z-factors leading to free theories as the parameter b goes to 0, the values of $\tilde{c}$ we obtain show that indeed these S-matrix theories have free bosons as their UV limits. This lends additional support to the conjecture that these S-matrices are those of real-coupling affine Toda field theories. But note that our methods can only confirm the general form of the Z-factors, not the dependence of their parameter b on the Toda coupling β (except that this dependence should be such that the Z-factors are nontrivial in the range of physical couplings).

In all of the above we are assuming of course, that the S-matrices under consideration are those of consistent QFTs. For most of the minimal S-matrix theories, for which no quantum field theoretic formulation is known at present, this is not obvious. For some perturbations of unitary CFTs, one can calculate certain correlation functions in the massive theory as perturbation series around the UV CFT, using special regularization techniques [59]. The fact that this perturbation expansion is expected [27, 60] to have a finite radius of convergence, provided one uses IR and – if necessary – UV cutoffs, might be used as heuristic evidence for the existence of the theory described by these correlation functions. Further investigation of the massive theories, identifying appropriate field-theoretic descriptions of them and calculating some of their Green functions, are important but very difficult tasks, in general. In principle, it is possible to extend the bootstrap program off shell once one knows certain qualitative features of the field content of the QFT underlying an S-matrix theory. One can then calculate matrix elements of local fields and find representations for the Green functions by inserting complete sets of asymptotic states (see e.g. ref. [61]). Only in the simplest cases, like the Ising field theory [61] of subsect. 3.1 and the models of subsect. 3.2 [23] have concrete results been obtained in this way. For the moment, we regard the consistency of the models considered in this paper as pure S-matrix theories (even at the multipole level, cf. sect. 2) and the agreement between the values of $\tilde{c}$ we calculated with those expected, as evidence for the existence of QFTs corresponding to these models.

Our results indicate that hopes raised in the literature for a direct relation between perturbed CFTs and real coupling ATFTs are unjustified. The basis for these hopes was the coincidence of the minimal S-matrices proposed for these theories. But, accepting the above conclusions, the S-matrices of the perturbed CFTs are *not* those of the ATFTs for special values of their (real) coupling. As was mentioned in sect. 1, arguments based on conformal field theory techniques suggest that perturbed CFTs are related to *imaginary* coupling ATFTs. In the case of the sine-Gordon model there is already a rather clear picture emerging of how certain restrictions (related to an underlying quantum group structure) in the soliton sector of the model, at special values of the coupling, lead to massive QFTs describing the $\phi_{(1,3)}$ perturbation of the Virasoro minimal models [23–25]. The S-matrices of these theories are correspondingly "restrictions" of the sine-Gordon model S-matrix at appropriate values of the coupling. They are not purely elastic, except for the perturbations of the nonunitary minimal models of subsect. 3.2, where the solitons are completely eliminated from the spectrum.

In fact, in any sense of the word, only "few" integrable perturbations of CFTs give rise to purely elastic scattering theories. Under certain perturbations some particles stay massless, and it is not known at present how to describe the scattering of the massive particles "dressed" with the massless ones. The best known examples are [2, 62, 63] the $\phi_{(1,3)}$-perturbations of unitary Virasoro minimal models, at least those high enough in the unitary series, in the direction that results in a renormalization group flow to the next lower model in the series. We mentioned earlier that the $\hat{\mathscr{G}}$-related S-matrix theories of sect. 3 (with the exception of the $A^{(2)}_{2n}$-related models) are obtained by specific perturbations of the first models in the $W(\mathscr{G})$ unitary series. Similar perturbations of the other unitary CFTs in these series are also known [6, 8] to have nontrivial IMs, whose spins are again conjectured to be the exponents of the corresponding Lie algebra. Besides the first model in each of these series, none of the others can be expected to have massive perturbations described by purely elastic S-matrices. The S-matrices of these latter models have been conjectured to be related to those of the first models by a "restriction", again coming from a quantum group symmetry [24].

There are also interesting questions that arise from our discussion of purely elastic scattering theories (sect. 3). Even though these theories are trivial from the point of view of the Yang–Baxter equation, they show a rich and beautiful structure. We think that some questions concerning multiple poles need further investigation in the context of pure S-matrix theory (i.e. not tied to a perturbative lagrangian framework as in the discussion of ATFTs). On a conceptual level, one would like to understand the relation of the minimal S-matrix theories to (affine) Lie algebras. Can one classify all purely elastic scattering theories, and if so, what is the precise connection (if any) to the classification of Lie algebras? We remark that so far no consistent S-matrices, purely elastic or otherwise, "related" to most of the non-simply laced (affine) Lie algebras have been found. A further study of the massive fusion rules, and their relation to the fusion rules of the corresponding CFT, should also be interesting.

Finally, we would like to discuss a conjecture which is suggested by some of the properties of the finite-size scaling coefficient $\tilde{c} = c - 12d_0$. Our conjecture is that $\tilde{c}$, if determined with doubly periodic boundary conditions, "measures" the number of massless degrees of freedom of an arbitrary two-dimensional (euclidean) QFT. Once we know more about the space of all such theories, we might want to suitably restrict the class of QFTs to be considered. We certainly want to include nonunitary (more precisely, non-reflection-positive) modular invariant, rational CFTs in this space, and it also seems natural to include *massive* non-reflection-positive QFTs. Massive euclidean field theories satisfying all physical requirements except reflection-positivity have been constructed nonperturbatively [64] using the Wilson renormalization group. For instance, ϕ^4-theory in four dimensions with *negative* coupling constant has a nontrivial, asymptotically free continuum limit, which most probably violates reflection-positivity [64]. In light of this example and the fact that the space of QFTs can be (locally) parametrized by the set of all coupling constants, it does indeed seem very natural to include non-reflection-positive theories, as apparently – at least in some cases – such theories can be obtained from reflection-positive ones by just reversing the sign of a coupling. (The fact that ϕ_4^4-theory with positive bare coupling presumably only allows for a trivial continuum limit is besides the point here).

Of course, since there is no a priori notion of some finite, positive number measuring the number of degrees of freedom of a QFT, there is no way we could actually prove our conjecture. The best one can hope for, is a generalization [65] of Zamolodchikov's c-theorem [62] to the abovementioned larger space of QFTs. Here we would just like to discuss some properties of $\tilde{c}$, which, we think, lend support to our conjecture. First of all, if the UV limit of the QFT is a unitary CFT, $\tilde{c}$ reduces to the central charge c, which was already proposed [62] to measure the number of degrees of freedom in this case. But $\tilde{c}$ has the right properties to do the job even when the UV limit is a nonunitary CFT. From our final result for $\tilde{c}$, eq. (67), we see that $\tilde{c}$ is manifestly positive. Furthermore, this equation allows us to associate a "partial" finite-size scaling coefficient $\tilde{c}_a$ to each particle in the massive scattering theory, apparently measuring its contribution to the total number of massless degrees of freedom of the theory. Again, $\tilde{c}_a$ seems to have the right properties to play this role: It is always between 0 and 1, 1 being the maximal number of degrees of freedom a massless particle (a boson) can have. In any given theory, $\tilde{c}_a$ is smaller the heavier the particle is, in almost all cases. This is expected intuitively, since one would think that a lighter particle has more massless degrees of freedom than a heavier one – at least if the two particles are not too different in other respects, e.g. in their interactions. Not surprisingly, the only particles for which this rule is violated are *the* fundamental particles in the $D_n^{(1)}$-related models, which are indeed quite different from the other particles in these models. Finally, the fact that $\tilde{c}_a$ is proportional to the entropy of a one-dimensional gas of free massless particles (with the chemical potential determined by the S-matrix of the theory) is also quite suggestive.

Up to now we considered eq. (67) that involves the S-matrix data of a massive perturbation of a CFT. Although one would hope for a generalization of this formula to an arbitrary perturbation of a CFT, our results at present apply only to perturbations leading to purely elastic scattering theories. But we can also discuss $\tilde{c} = c - 12d_0$ directly in terms of CFT data. One would like to show – at least for modular invariant, rational CFTs – that $c - 12d_0$ is positive, and smaller than the value of c of the free bosons and fermions used in a free-field representation [13, 14] of the given CFT. It would also be very nice if the trivial theory turns out to be the unique modular invariant, rational CFT with $c - 12d_0 = 0$. There are good reasons [65] to believe that $c - 12d_0 > 0$ for all nontrivial rational CFTs on the torus. As an example, consider the Virasoro minimal models, where a complete classification of modular invariant partition functions exists [66] for both unitary and nonunitary models. Then one can verify [66] that the spinless field whose left and right dimensions equal the minimal weight in the Kac table always appears in the model. Accordingly, for *any* modular invariant theory with labels p' and p, one has $c - 12d_0 = 1 - 6/pp'$. As p' and p are mutually prime and both greater than 1, we see that $c - 12d_0 \geqslant 0$, with an equality only for the trivial theory for which p' and p are 2 and 3. Note also that $c - 12d_0 < 1$, as expected from the above discussion, as all the Virasoro minimal models can be constructed [13] using a single free boson.

We believe that a generalization of eq. (67) for $\tilde{c}$ to an arbitrary factorizable S-matrix theory could be an important tool in the study of integrable massive two-dimensional quantum field theories. Furthermore, investigating the general properties of the finite-size scaling function $\tilde{c}(r)$ and the finite-size scaling coefficient $\tilde{c}$ should provide new insights about the *space* of two-dimensional quantum field theories.

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