The Bologna Lectures on Entanglement and Non-Equilibrium Physics in Extended Quantum Systems

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> November 2016 University of Bologna

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Part A: Introduction to integrable QFT and CFT, and non-equilibrium steady states.

by Benjamin Doyon

Contents

1	Intr	roduction	
	Inte	$grability, \ locality, \ the \ scaling \ limit \ (1h)$	3
	1.1	Liouville integrability	3
	1.2	Locality, and quantum integrability	5
	1.3	The scaling limit	7
	1.4	Integrable QFT	9
2	Bas	ics of CFT	
	Chi	ral factorization, conformal Ward identities (1h)	11
	2.1	Critical points and chiral factorization	11
		2.1.1 Noether's theorem from locality and translation invariance	12
		2.1.2 Lorentz invariance	13
		2.1.3 Scale invariance	13
		2.1.4 Chiral factorization	14
	2.2	Conformal Ward identities and conformal invariance	14
3	Bas	ics of integrable QFT	
	Fact	torized scattering from local conserved quantities (1h)	19
	3.1	Asymptotic states in massive models	19
	3.2	Elastic scattering	21
	3.3	Factorized scattering: Yang-Baxter equations	23
	3.4	Analytic and other properties of the two-particle S -matrix	26

	3.5	mple examples	29
		5.1 Recapitulation of the requirements	29
		5.2 A diagonal example	29
		5.3 A non-diagonal example	30
	3.6	dditional remarks	31
4	Nor	quilibrium quantum steady states	
	Cur	its and their fluctuations at criticality (2h)	37
5	Hyd	dynamic approaches to non-equilibrium steady states	
	Gen	alized hydrodynamics (3h)	37

General introduction to the course

Integrability is a wide subject that comprises many deep ideas and that can be applied to very diverse physical systems. The most developed theories of integrability occur in the contexts of finite dynamical systems, classical continuous media, quantum lattices, and quantum field theory (QFT). This course will give an overview of modern applications of integrability to physical quantities and situations of high current interest, concentrating on QFT. The two areas of applications covered are that of entanglement measures and that of non-equilibrium physics, in both cases as they occur in extended quantum systems. Extended quantum systems, and in particular the emergent behaviours they display, give rise to some of the most interesting physical phenomena as well as to the most difficult theoretical challenges. QFT, and in particular conformal field theory (CFT), is a powerful theory for such emergent phenomena, and when combined with integrability it offers a unique opportunity for exact and nontrivial predictions. The application to entanglement measures and to non-equilibrium systems constitute, in our opinion, some of the most striking applications of integrability QFT, giving rise to some of the most far-reaching predictions.

There has been a very large amount of activity in these areas in recent years, and this course will cover a small part of each area. The choice of topics is entirely dictated by our research interest and results, and we do not attempt to provide an exhaustive overview of any of these subjects. Nevertheless, we hope that the topics covered will give an intuition as to the problems and ideas that arise in these areas of studies. We also provide an overview of the literature that will point to some of the aspects that were not covered.

The course is divided into two parts: Part A: Introduction to integrable QFT and CFT, and non-equilibrium steady states will be covered by Benjamin Doyon, and Part B: Approaches to integrable QFT, and measures of entanglement will be covered by Olalla A. Castro Alvaredo. The two parts will be intertwined: we will first develop the general theory of integrable QFT and CFT, before studying their applications. These notes only cover Part A.

The material in the present notes is strongly based on the following sources: (1) my "Introduction to Integrable Quantum Field Theory" lecture notes from Oxford (2006) and Durham (2008); (2) my "Integrability" lecture notes for the London Taught Course Centre (2012); and (3) my review written in collaboration with Denis Bernard: "Conformal field theory out of equilibrium: a review", J. Stat. Mech. 2016, 064005 (2016), some part of which were themselves based on my lectures "Steady states in extended quantum systems out of equilibrium: currents and fluctuations" given at the Nancy Thematic School 2015 Introduction to Out-of-equilibrium Quantum Many-Body Physics.

See:

http://benjamindoyon.weebly.com/lecture-notes.html

https://arxiv.org/abs/1603.07765

This is a preliminary version, containing only the first three chapters of the notes

1 Introduction

Integrability, locality, the scaling limit (1h)

1.1 Liouville integrability

In physics, what is usually understood as a *system* is a topological space of states M (usually a manifold, although often infinite-dimensional), and an evolution map, a bijective map U_t : $M \to M$ parametrised by the time $t \in \mathbb{R}$. Every state is a mathematical object which encodes observable predictions, and the evolution map determines how the predictions change in time. In various physical applications, the space of states may be very different, hence also the evolution equation may look different. In classical dynamical systems, the space of states is the phase space: a symplectic manifold (hence with a non-degenerate Poisson structure). In classical field theory, it is a space of functions (often on on \mathbb{R}). In quantum mechanics and in quantum field theory, it is a Hilbert space (but in both cases with somewhat different structures).

In most, if not all, physical cases, the evolution equation is such that time evolution of any given state is continuous: the map $t \mapsto U_t(p)$, for any given $p \in M$, is a continuous map (at least in a neighbourhood of t = 0). However, the map U_t from M to M, for fixed time t, is in general very complicated. In particular, although it may be almost everywhere continuous, this continuity is almost nowhere uniform in time. For instance, in many cases, two nearby states map to states that are very far apart, and that become exponentially further apart as t is increased – this is chaos. Integrability is essentially the opposite of chaos: the map U_t is as nice as it can be. It possesses infinitely many invariant submanifolds that foliate M, parametrised by as many continuous parameters as there are "degrees of freedom", and on these submanifolds, states that start nearby stay nearby uniformly in time. This is what is explicitly seen in classical dynamical systems, and it seems to form a common underlying principle at the basis of integrability in general.

Consider a classical system. First, as mentioned, this is a symplectic 2*n*-dimensional manifold, and for simplicity we will take it to be described by the usual coordinates $p_i \in \mathbb{R}$, $q_i \in \mathbb{R}$: i = 1, ..., n with canonical Poisson brackets

$$\{p_i, q_j\} = \delta_{i,j}.$$

Let us denote by \mathcal{F} the real linear space of smooth real functions on our symplectic manifold, which of course also has the structure of a ring. We have, as usual, for any functions $F, G \in \mathcal{F}$,

$$\{F,G\} = \sum_{i} \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \right).$$

Second, the classical system is endowed with equations of motion. These are equations representing the evolution along the vector field associated to a particular function on phase space, the Hamiltonian H = H(p, q). That is, the equations of motion are the statement that

for any function F on phase space¹,

$$\dot{F} = \{H, F\}\tag{1}$$

(where the dot means time derivative d/dt).

The definition of Liouville integrability is as follows.

Definition 1.1 A dynamical system (of 2n-dimensional phase space) is Liouville integrable if there exists n independent conserved quantity in involution.

This definition requires explanation for the various terms involved. A conserved quantity is a function $F \in \mathcal{F}$ that is invariant under time evolution,

$$\{H,F\} = 0.$$

We require there to be n of them, F_i , i = 1, ..., n. These are required to be *independent*. This means that at generic points on the symplectic manifold, the tangent space of the surface defined by $F_i = f_i = const. \forall i$ exists and is n-dimensional (hence, these relations give rise, locally, to a n-dimensional submanifold); or equivalently, that the total variations dF_i are linearly independent. Finally, the conserved quantities are in *involution* if

$$\{F_i, F_j\} = 0 \quad \forall \quad i, j.$$

An important remark is that there cannot be more than n independent conserved quantities in involution. This remark implies that the Hamiltonian itself, which is obviously conserved and in involution with all of F_i , must be a function of the F_i (hence it can always be taken as one of them). In fact, a Liouville integrable system is not really characterized by its Hamiltonian, but rather by its set of quantities in involution (modulo non-singular transformations amongst them); the choice of a Hamiltonian as a function of these is essentially irrelevant.

Theorem 1.2 (the Liouville theorem) The solution of the equations of motion of a Liouville integrable system is obtained by "quadrature".

By quadrature means that we only have to solve some algebraic equations and do some integrals in order to obtain the solution. That this be the case is certainly not immediate from the definition of a dynamical system in general.

Remark 1.1 Let us assume that at every point (not just "generically"), dF_i are linearly independent. Then we have an *n*-dimensional submanifold M_f defined by fixing the conserved quantities F_i to constants, $M_f = \{(p,q) : F_i = f_i\}$. Since all conserved quantities are in involution, we can define a map $m_t : M_f \to M_f, (p,q) \mapsto (p(t),q(t))$ for every $t \in \mathbb{R}^n$ by solving

$$\frac{d}{dt_1}\cdots\frac{d}{dt_n}G = \{F_1, \{\cdots, \{F_n, G\}\cdots\}\}$$

(for any $G \in \mathcal{F}$), and this gives rise to a transitive and locally free action of the abelian group \mathbb{R}^n on \mathbb{M}_f ,

$$m_t \circ m_{t'} = m_{t+t'}.$$

Let us assume further that M_f is compact and connected. This means that for any point on the manifold, there will be a discrete subgroup of \mathbb{R}^n that fixes it, which must be isomorphic to \mathbb{Z}^n (because there will be *n* independent directions in \mathbb{R}^n in which the point is fixed). Hence, the manifold has the quotient

¹We consider here only dynamical systems with explicitly time-independent Hamiltonians.

structure $\mathbb{R}^n/\mathbb{Z}^n$, which is topologically (and from the viewpoint of the differentiable manifold structure) a (multi-dimensional) torus. These are the invariant tori of integrable dynamical systems.

The general form of the solution may then be obtained by transforming to the action-angle variables. Since M_f are tori, there are *n* cycles. Let us denote by c_j for j = 1, ..., n the *n* cycles on M_f . Liouville integrability implies that we can define the so-called action-angle variables. Angle variables θ_j describe paths along these cycles,

$$\int_{c_j} d\theta_k = \delta_{k,j}$$

and action variables I_j are canonical conjugate,

$$\{I_j, \theta_k\} = \delta_{j,k},\tag{2}$$

which are purely functions of the conserved quantities $I = I(\{F_j\})$, hence are invariant under time evolution. This implies that angles evolve linearly with time:

$$\{H, \theta_j\} = \frac{\partial H}{\partial I_j} \Rightarrow \theta_j(t) = \frac{\partial H}{\partial I_j} t + \theta_j(0)$$
(3)

and thus the full solution is extremely simple in action-angle variables.

1.2 Locality, and quantum integrability

How is the concept of integrability modified in quantum systems?

For definiteness, consider the Heisenberg quantum chain with N sites. The Hilbert space is a tensor product (over \mathbb{C}) of N spin-1/2 Hilbert spaces, $\mathcal{H}_N = (\mathbb{C}^2)^{\otimes N}$, and the Hamiltonian has the simple form

$$H = \sum_{n=1}^{N} h_n, \qquad h_n := \vec{\sigma}_n \cdot \vec{\sigma}_{n+1}.$$
 (4)

Here, $\vec{\sigma}_j$ is the vector of Pauli matrices σ_n^x , σ_n^y , σ_n^z , and the index j indicates that the operator acts nontrivially on site j and is trivial (the identity) on other sites, e.g.

$$\sigma_n^{\rm x} = \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \sigma^{\rm x} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$$

For simplicity, we will restrict ourselves to the case where the chain is periodic, and make the identification $\vec{\sigma}_{n+1} \equiv \vec{\sigma}_n$. Note that the total-spin operator

$$\vec{S} := \frac{1}{2} \sum_{n=1}^{N} \vec{\sigma}_n \tag{5}$$

commutes with the Hamiltonian, $[H, \vec{S}] = 0$, reflecting the sl(2)-symmetry of the model. We will use this model and some simply generalizations in order to illustrate various concepts in these notes.

Our first task is to define integrability. We may wish to follow the classical case, and ask for as many integral of motions as there are degrees of freedom. The problem is that for any finite N, the Hamiltonian (4) is just a finite-dimensional Hermitian matrix 2^N by 2^N , hence it can be diagonalized. In the diagonal basis, any other matrix that is diagonal will commute with the Hamiltonian. There are exactly 2^N such other, independent matrices, and they all commute with each other. Hence, we have automatically 2^N commuting conserved quantities in involution. Does that mean that any quantum chain is integrable – in fact, so integrable that instead of the N conserved quantities we could expect for N degrees of freedom, we have 2^N of them? Of course not. There is no universally accepted, simple definition of integrability for quantum spin chains. However, one of the main concepts that is commonly accepted as being fundamental is that of *locality*.

Locality in quantum spin chains makes sense only when considering the *thermodynamic* limit $N \to \infty$. We think of the thermodynamic limit as the process by which the space \mathcal{H}_N is augmented, with inclusion $\mathcal{H}_N \subset \mathcal{H}_{N+1}$ in the natural way (this is the inductive limit). We make three different, but related, uses of the epithet "local", for three related types of objects.

First, we introduce the concept of support: the support of an operator is the set of sites on which it is nontrivial. By definition, on any site away from the support, the operator factorizes into the identity operator, which we may express by the fact that

$$[\mathcal{O}, A_n] = 0$$
 for every $A_n = \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes A \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$ and *n* not in the support of \mathcal{O} . (6)

For instance, the operator h_1 is supported on the sites 1 and 2, and the operator \vec{S} on the whole chain.

Second, we say that the quantum spin chain model is *local* if the Hamiltonian H, as the thermodynamic limit $N \to \infty$ is taken, is always of the form $\sum_n h_n$, and there exists a r > 0 such that for every n, the operator h_n is supported on sites lying within the interval [n-r, n+r]. It is important that r does not depend on N: this is the main point of a local quantum spin chain model. Clearly, this is the case of the Heisenberg Hamiltonian, taking $N \to \infty$ in (4).

Third, we say that an operator \mathcal{O}_n is *local* around the site *n* if the following holds: \mathcal{O}_n commutes with the Hamiltonian density h_m whenever |n - m| is large enough. That is:

$$\exists r > 0 : [\mathcal{O}_n, h_m] = 0 \quad \forall m : |n - m| > r.$$

$$\tag{7}$$

If \mathcal{O} is local around 0 and its translates \mathcal{O}_n are local around n, then we simply say that \mathcal{O} , or \mathcal{O}_n , is a local operator. When the quantum chain is homogeneous, locality is used in this way, with translates being local around n. We also say that two operators \mathcal{O} and \mathcal{O}' are *local with* respect to each other, or mutually local, if $[\mathcal{O}_n, \mathcal{O}'_m] = 0$ for |n - m| large enough. Clearly, an operator \mathcal{O} that is supported on a bounded number of sites is a local operator, and any two such operators are local with respect to each other; the latter is often taken as a (more restrictive) definition of a local operator, but in many situations, for instance when studying entanglement entropy, the more general definition above is necessary.

Example 1.1 Consider the operator $u_n = e^{i\vec{v}\cdot\vec{\sigma}_n}$. Clearly this is supported on n, hence is local. Note that $u_n\vec{\sigma}u_n^{\dagger} = R\vec{\sigma}$ where R is the rotation matrix about the axis in the direction \vec{v} and by an angle $|\vec{v}|$. Now consider the operator $\mathcal{T}_n = \prod_{m \ge n+1} u_m$. Clearly this has infinite support. It satisfies the following exchange relations:

$$\mathcal{T}_n \vec{\sigma}_m = \begin{cases} R \vec{\sigma}_m \, \mathcal{T}_n & (m \ge n+1) \\ \vec{\sigma}_m \, \mathcal{T}_n & (m \le n). \end{cases}$$
(8)

This means that it "acts as" a local spin rotation on sites $m \ge n+1$. As a consequence, since $(R\vec{\sigma}_m) \cdot (R\vec{\sigma}_{m+1}) = \vec{\sigma}_m \cdot \vec{\sigma}_{m+1}$, we have $\mathcal{T}_n h_m = h_m \mathcal{T}_n$ for all $m \ge n+1$ and $m \le n-1$. That is, although \mathcal{T}_n has infinite support, it is a local operator according to our definition. \mathcal{T}_n is a twist operator associated to the symmetry generated by \vec{S} : it is the exponential of "half the sum" that defines S, see (5). The branch-point twist field, that will be discussed in Part B of the lectures, is an operator of this type.

Finally, we say that a *local conserved quantity* (or local conserved charge) is an operator Q supported on the whole chain, such that [Q, H] = 0 (at least when taking the thermodynamic

limit, within an appropriate topology which we do not specify here), and such that Q is a sum over n of uniformly local operators around n:

$$Q = \sum_{n} q_{n}, \qquad \exists r > 0 : [q_{n}, h_{m}] = 0 \ \forall |n - m| > r.$$
(9)

Using these concepts, our notion of integrability is as follows:

Definition 1.3 A local quantum spin chain model is integrable if there exists infinitely many local conserved quantities $Q^{(k)}$ that are in involution, $[Q^{(k)}, Q^{(l)}] = 0 \forall k, l$, and whose densities are local with respect to each other.

Like in the case of Liouville integrable systems, clearly H is itself one of the infinitely many conserved quantities. In this definition, we do not specify "how big" the infinity must be; this is a delicate question which we will not address (and to which there is no commonly accepted and precise answer yet²). The Heisenberg chain is integrable according to the above definition. Indeed it is possible to show that there are infinitely-many local conserved quantities. For instance, it is simple to check that the following is one of them:

$$Q = \sum_{n=1}^{N} j_n, \qquad j_n := 2(\vec{\sigma}_{n-1} \times \vec{\sigma}_n) \cdot \vec{\sigma}_{n+1}.$$
 (10)

This gives a formal definition of integrability, but does not explain how integrability leads to solvability. What replaces the classical concept of solution by quadrature in quantum models? This is the subject of much developments in integrable quantum systems, and we will describe in some details how this works in integrable QFT.

1.3 The scaling limit

In all physical applications of integrable QFT that we will develop in this course, it is paramount to understand how QFT is related to quantum chains. Consider the so-called "XXZ" Heisenberg spin chain: the Hamiltonian is a slight generalization of the Heisenberg spin chain, still integrable,

$$H = \sum_{n} \left(\sigma_n^{\mathbf{x}} \sigma_{n+1}^{\mathbf{x}} + \sigma_n^{\mathbf{y}} \sigma_{n+1}^{\mathbf{y}} + \Delta \sigma_n^{\mathbf{z}} \sigma_{n+1}^{\mathbf{z}} \right).$$

The parameter Δ is an anisotropy: the z direction of the magnetic interaction is different.

Many of the quantities of physical interest are obtained from *correlation functions*: averages, within the ground state, of products of observables of local operators at different sites of the chain. For $\Delta > 1$, complicated calculations using integrability show that correlation functions for two separated sites factorize with exponential accuracy at large separations, for instance:

$$\langle \sigma_n^{\mathbf{x}} \sigma_m^{\mathbf{x}} \rangle - \langle \sigma_n^{\mathbf{x}} \rangle \langle \sigma_m^{\mathbf{x}} \rangle \propto e^{-\frac{|n-m|}{\xi}} \quad \text{as} \quad |n-m| \to \infty \qquad (\Delta > 1).$$

for some number ξ . The same holds for any combination of $\sigma_n^{x,y,z}$ and $\sigma_m^{x,y,z}$, as well as for any local operators at sites n and m, potentially for different values of ξ . Hence if we look at very large distances, we see no quantum correlations at all; this seems natural, since the interaction

²One may propose an answer for the opposite (non-integrability) within the framework of C-star algebras, see the concept of a completely mixing Hamiltonian in B. Doyon, Thermalization and pseudolocality in extended quantum systems, arXiv:1512.03713

is just between nearest neighbors. We define the largest number ξ that appear as the *correlation* length. This correlation length is a function of the parameter Δ , i.e. $\xi = \xi(\Delta)$.

In general, $\xi(\Delta)$ is related to the structure of the spectrum of the Hamitonian H: if, in the thermodynamic limit, there remains a finite gap between the energy of the ground state and that of the first excited state, then $\xi(\Delta)$ is nonzero, and proportional to the inverse of this gap.

In the XXZ chain, the exact expression of $\xi(\Delta)$ can be obtained. This expression shows that the correlation length tends towards infinity as $\Delta \to 1^+$. Since correlation functions summed over all sites give rise to *response functions*, this will imply singularities in the magnetic susceptibility, for instance, so that this is a second order (quantum) phase transition. Exactly at $\Delta = 1$, we have for instance

$$\langle \sigma_n^{\mathbf{x}} \sigma_m^{\mathbf{x}} \rangle \propto |n - m|^{-2d} \quad \text{as} \quad |n - m| \to \infty \qquad (\Delta = 1)$$

where d is some positive number (known exactly) that depends on the observable σ_n^x that we have chosen (the product of one-point functions are not necessary because they are zero by symmetry). The point $\Delta = 1$ is a quantum critical point of the family of XXZ models.

The scaling limit is obtained by looking at very large distances |j-k| while at the same time taking the limit $\Delta \to 1^+$. In general, take any correlation function of local operators $\mathcal{O}_{n_1}^{(1)}, \mathcal{O}_{n_2}^{(2)}$ at positions n_1, n_2, \ldots Then take the limit where all positions n_1, n_2, \ldots are scaled up while at the same time the correlation length is sent to infinite (i.e. $\Delta \to 1^+$), with constant ratios $n_1/\xi, n_2/\xi, \ldots$ This limit as such is zero; but now take the same limit, where the correlation function is multiplied by $\xi^{d_1}\xi^{d_2}\cdots$ for appropriate constants d_1, d_2, \ldots that depend on which operators are at positions n_1, n_2, \ldots respectively³. Then the result of this limit is nonzero. This is the continuum or *scaling limit*, and the result of the limit is described by QFT. The result is largely independent of the details of the underlying quantum chain: different quantum chains will give rise to the same limits of correlation functions. This is universality of the scaling limit: models fall into universality classes. QFT is a theory describing the universality classes.

In equation, for instance for the *n*-point function of σ_n^x , we set $\ell = \ell(\Delta) = m\xi(\Delta)$ for some (dimensionful) quantity *m*, and take the limit

$$\lim_{\Delta \to 1^+} \ell^{nd} \langle \sigma_{\ell x_1}^{\mathbf{x}} \cdots \sigma_{\ell x_n}^{\mathbf{x}} \rangle \propto \langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle_{\text{QFT}}$$
(11)

where the limit is taken keeping the (dimensionful) parameters x_1 and x_2 fixed. On the righthand side, we have a correlation function in a QFT with smallest mass equal to m, and the field $\mathcal{O}(x)$ at position x is now an appropriate local field (see below) "representing", in the QFT, the local operator σ_n^x . The unique (positive) number d making the limit finite is called the *scaling dimension* of the field $\mathcal{O}(x)$. The QFT correlation function is a referred to as a scaling function. In this case, the QFT is integrable: it is the quantum sine-Gordon model (at a very special point, equivalent to a free massless boson times the SU(2)-Thirring model).

The scaling limit may be taken directly at criticality $\Delta = 1$ (i.e. at $\xi = \infty$). For instance, for *n*-point functions,

$$\lim_{\ell \to \infty} \ell^{nd} \langle \sigma_{\ell x_1}^{\mathbf{x}} \cdots \sigma_{\ell x_n}^{\mathbf{x}} \rangle |_{\Delta=1} \propto \langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_2) \rangle_{\text{CFT}}.$$
 (12)

This gives rise to a conformal field theory (CFT), as we will explain in the next chapter. Note that in most cases (but not in all!), the same result is obtained by taking the limit $mx_1, mx_2 \rightarrow 0$

³In fact, more generally, one has to choose families of local operators parametrized by ξ , corresponding to general renormalization; here we are choosing the families ($\xi^{2d_k} \mathcal{O}_{n_k}^{(k)} : \xi > 0$), corresponding to multiplicative renormalization.

of the massive QFT correlation function – that is, the massless limit, or equivalently the shortdistance limit, of QFT correlation functions. Note the dichotomy: a massive QFT at *short distances* gives rise to a CFT, while a critical quantum chain at *large distances* gives rise to a CFT.

We make three additional notes:

- The proportionality constants in (11) and (12), independent of the distances, are not determined by the QFT or the quantum chains. They relate to the normalizations of the QFT fields, and are not universal. Hence, the universal information in the QFT correlation function is the distance dependence. However, the proportionality constants usually are of the form of a constant to the power n in an n-point function. Therefore appropriate ratios of correlation functions are fully universal.
- The scaling limit equations (11) and (12) tell us that QFT and CFT correlation functions describe *asymptotic behaviours* of quantum chain correlation functions; either at large correlation length and large distances simultaneously (massive QFT), or at infinite correlation length and then large distances (CFT).
- One may also take the scaling limit in other states than the ground state. For instance, one may use a thermal state, characterized by the temperature T. The scaling limit is then the limit where simultaneously the distances and correlation length are taken large, and also the thermal length T^{-1} is taken large, all with constant ratios.

1.4 Integrable QFT

There are two related ways of defining a QFT model. One is based on the local-field picture, the other on the scattering picture. The relation between these two is nontrivial, and, as we will see, is what the integrability structure of QFT allows us to make more explicit.

First, a QFT model will be completely specified (hence all physical information can be extracted, at least in principle) once we have the following ingredient. The basic ingredients are:

- Hilbert space: a Hilbert space \mathcal{H} .
- Hamiltonian: an operator H acting on \mathcal{H} , called the Hamiltonian, which is diagonalizable and whose eigenvalues are bounded from below; the vector associated with the lowest eigenvalue is the vacuum, $|vac\rangle \in \mathcal{H}$, and the lowest eigenvalue is set to 0.
- **Relativistic invariance:** operators *P* (the momentum operator) and *B* (the boost operator) which satisfy

$$[H, P] = 0 , \quad [B, P] = H , \quad [B, H] = P$$
(13)

as well as

$$H|\mathrm{vac}\rangle = P|\mathrm{vac}\rangle = B|\mathrm{vac}\rangle = 0.$$
 (14)

But his is not enough for defining a QFT model. There are two ways of completing the definition: adding what we mean by the local observables, or adding the information of the scattering between the particles. These two ways are equivalent.

In the locality picture, we add the following condition:

• Locality: there exist h(x) and p(x) such that

$$H = \int \mathrm{d}x \, h(x) \,, \quad P = \int \mathrm{d}x \, p(x) \tag{15}$$

and such that h(x) and p(x) are local fields.

Here, following the quantum chain definition, we say that an x-dependent operator $\mathcal{O}(x)$ is a local field if $[\mathcal{O}(x), h(x')] = 0$ for $x \neq x'$, and if additionally

$$[P, \mathcal{O}(x)] = i \frac{d}{dx} \mathcal{O}(x)$$
(16)

(this identifying x as a position). We also locality of local fields $\mathcal{O}(x)$ and $\mathcal{O}'(x)$ with respect to each other as

$$[\mathcal{O}(x), \mathcal{O}'(x')] = 0 \quad \forall \ x \neq x', \tag{17}$$

and local conserved quantities similarly to the case of local quantum chains:

$$Q = \int \mathrm{d}x \, q(x), \qquad q(x) \text{ a local field.}$$
(18)

This then allows us to define integrable QFT in exactly the same way that integrability was defined for quantum chains:

Definition 1.4 A QFT is integrable if there exists infinitely many local conserved quantities $Q^{(k)}$ that are in involution, $[Q^{(k)}, Q^{(l)}] = 0 \forall k, l$, and whose densities are local with respect to each other.

In the scattering picture, the conditions to add in order to define a QFT are more involved, and relate to analytic properties of the scattering matrix. We discuss some of these aspects in Section 3. First, though, we develop basic aspects of CFT, which are completely based on the locality concepts discussed above.

– Exercises ——

Exercise 1.1 Prove that I as defined in (10) commutes with H as defined in (4).

— Bibliographic notes —

There are very many good references on classical integrable systems, and since this is such an old and large subject, we only refer here to a recent book covering parts of the subject:

"Introduction to classical integrable systems", O. Babelon, D. Bernard and M. Talon, Cambridge University Press, 2003

The theory of integrability in quantum chains essentially started with the work of Hans Bethe on the Heisenberg chain,

H. Bethe, "On the Theory of Metals", Z.Phys 71, 205 (1931).

Important developments were made by Yang and Yang, in this series of papers:

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2 Basics of CFT

Chiral factorization, conformal Ward identities (1h)

2.1 Critical points and chiral factorization

Quantum lattice models at criticality display very special low-energy behaviours. The main property of criticality is scale invariance: at critical points, physical quantities involving only states of low energies above the ground states are scale invariant or co-variant. Such physical quantities are correlation functions at large distances like those discussed in the previous section, since only low-energy excitations can travel large distances in order to generate correlations. Combined with other physically motivated (and often observed) symmetries, this leads to powerful predictions for these low-energy behaviours.

The assumptions made are that at in the scaling limit the following three groups of symmetries emerge:

(i) continuous translation invariance,

- (ii) Lorentz invariance, and
- (iii) scale transformation invariance.

First, emergence of continuous translation invariance is relatively easy to understand: the quantum chain is translation invariant under site translations, and at large distances, one site looks infinitesimal; thus if the large-distance limit exists, it must display continuous translation invariance (a proper understanding would involve arguing for the slow-varying structure of low-energy excitations above the ground state).

Remark 2.1 We may then argue as follows for the "momentum part" of the local QFT structure to indeed emerge in the scaling limit. A consequence of translation invariance is the existence of a hermitian operator P such that (13), (14) and (16) hold. By homogeneity, we must be able to write P as in (15) for some $p(x) = e^{-iPx} p(0) e^{iPx}$. By virtue of (16), it must be possible to choose p(x) so that it be local: it is the local operator whose commutator with $\mathcal{O}(x')$ is supported at x = x' with weight $i\partial_x \mathcal{O}(x)$.

Second, Lorentz invariance is here assumed to emerge – this is the case in the Heisenberg chain discussed above for instance. It does so whenever *time* must scale exactly like distances in order for the scaling limit of time-dependent correlation functions to be finite. This occurs, for instance, whenever the dispersion relation for low-energy excitations is linear (such as low-energy excitations above a Fermi sea, in one dimension).

Finally, scale invariance is simply a consequence of the correlation length being infinite. In (12) for instance, one may always rescale $\ell \mapsto \lambda \ell$ and the same limit is obtained, up to a constant λ^{2d} .

Let us now analyze some immediate consequences of these invariances.

2.1.1 Noether's theorem from locality and translation invariance

In QFT on \mathbb{R} , we often think of local fields at infinity to be zero, or at least to have vanishing influence on whatever physical quantity we are evaluating. This is not always the case, as nontrivial asymptotic conditions may give nonzero values to fields at infinity. However, for the sake of studying conserved charges, this is usually a good assumption. A rough argument is that a conserved quantity I is to be used either as an evolution operator e^{itI} or in the construction of density matrix $e^{-\beta I}$, and in both cases, clustering at large distances guarantee that the form of I very far from any local field $\mathcal{O}(x)$ (say in the time-evolved field $e^{itI}\mathcal{O}(x)e^{-itI}$ or average $\text{Tr}(e^{-\beta I}\mathcal{O}(x))/\text{Tr}(e^{-\beta I})$) is vanishingly small. This means that if a local conserved quantity $I = \int dx \,\rho(x)$ is in fact identically zero, I = 0, then we may see this as implying that $\rho(x) = \partial_x(\cdots)$ where \cdots is a local field. This leads to Noether's theorem: let $I = \int dx \,\rho(x)$ be a local conserved quantity, not identically zero. Then [H, I] is an identically-zero local conserved quantity with density $[H, \rho(x)] = -i\partial_t \rho(x)$, and we conclude the existence of a local field $j_\rho(x)$ such that

$$\partial_t \rho(x) + \partial_x j_\rho(x) = 0.$$

That is, there is a local conserved current (a Noether current).

Now we use this for conservation of H and P. We find that there must exists a local observable j(x) such that

$$\partial_t h + \partial_x j = 0 \tag{19}$$

and that there must exist a local observable k(x) such that

$$\partial_t p + \partial_x k = 0. \tag{20}$$

Therefore, from continuous translation invariance near a critical point we have obtained the existence of densities h(x) and p(x) and of the associated currents j(x) (the energy current) and k(x) (the momentum current – which is essentially the pressure) with the conservation equations (19) and (20). These can be put into the stress-energy tensor,

$$T^{\mu\nu} = \left(\begin{array}{cc} h & j \\ p & k \end{array}\right)^{\mu\nu} \tag{21}$$

and we have $\partial_{\nu}T^{\mu\nu} = 0$.

2.1.2 Lorentz invariance

The boost operator generates, at t = 0, infinitesimal Lorentz transformations, obtained on spinless fields by the action of the operator $x\partial_t$. Since ∂_t is generated by the Hamiltonian, it is natural to expect (and usually is the case) that the boost operator takes the form

$$B = -i \int dx \, x h(x). \tag{22}$$

This must satisfy the relations

$$[B, H] = P, \quad [B, P] = H. \tag{23}$$

The second relation is a direct consequence of (16), so does not impose any condition. The first relation, in combination with (19), then implies $P = \int dx j(x)$, and thus, since this is an operator relation, leads to the equality between the energy current and the momentum density (up to total derivative terms that can be absorbed into appropriate redefinitions):

$$j = p. \tag{24}$$

That is, $T^{\mu\nu} = T^{\nu\mu}$.

2.1.3 Scale invariance

We may take a similar approach as above: we require that the operator

$$D = -i \int dx \, x p(x), \tag{25}$$

which generates infinitesimal scale transformations $x\partial_x$ (again, here at t=0), satisfies

$$[D, H] = H, \quad [D, P] = P.$$
 (26)

Again, the second relation is immediate, and the first implies

$$k = h. (27)$$

That is, $T^{\mu}_{\mu} = 0$ (recall the Minkowski metric).

2.1.4 Chiral factrorization

We now combine (19), (20), (24) and (27). The result is strikingly simple: the two conservation equations

$$\partial_t h + \partial_x p = 0, \quad \partial_t p + \partial_x h = 0 \tag{28}$$

can be combined into "chiral factorization":

$$\bar{\partial}T = 0, \quad \partial\bar{T} = 0$$
 (29)

where

$$T \propto \frac{h+p}{2}, \quad \bar{T} \propto \frac{h-p}{2}, \quad \partial = \frac{1}{2}(\partial_x - \partial_t), \quad \bar{\partial} = \frac{1}{2}(\partial_x + \partial_t).$$
 (30)

Thus the quantum time evolution is

$$T(x,t) = T(x-t), \quad \bar{T}(x,t) = \bar{T}(x+t)$$
 (31)

(where here and below, a single argument means that the operator is evaluated at time t = 0). As a consequence of (31), in any state $\langle \cdots \rangle$ that is clustering at large distances, and invariant under time evolution, factorization occurs:

$$\langle \prod_{i} T(x_i) \prod_{j} \bar{T}(y_j) \rangle = \langle \prod_{i} T(x_i) \rangle \langle \prod_{j} \bar{T}(y_j) \rangle.$$
(32)

Indeed, we can just evolve in time the left-hand side for an infinite time and use invariance and clustering to obtain the right-hand side. This is chiral factorization.

We can rephrase this in Euclidean time $\tau = it$. Using coordinates $z = x - t = x + i\tau$ and $\bar{z} = x + t = x - i\tau$, the three conditions give rise to

$$T^{z\overline{z}} = T^{\overline{z}z} = 0, \quad \partial_z T^{zz} = 0, \quad \partial_{\overline{z}} T^{\overline{z}\overline{z}} = 0$$
(33)

and we define $T \propto T^{\bar{z}\bar{z}}$ and $\bar{T} \propto T^{zz}$, obtaining

$$T = T(z)$$
 holomorphic, $\overline{T} = \overline{T}(\overline{z})$ anti-holomorphic. (34)

2.2 Conformal Ward identities and conformal invariance

Above we saw how translation, Lorentz and scale invariance lead, in one dimension, to chiral factorization, or equivalently holomorphicity and anti-holomorphicity of appropriate components of the stress-energy tensor. However, according to general Ward-Takahashi identies, the current conservation equations at the basis of this only hold, within correlation functions, at positions away from that of other fields.

In general, for a space-time symmetry transformation $x \mapsto g(x)$, with action on fields $\mathcal{O}(x) \mapsto g[\mathcal{O}(x)]$, we have

$$\langle g[\mathcal{O}(x_1)] \cdots g[\mathcal{O}(x_n)] \rangle_{g(D)} = \langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle_D.$$
 (35)

Here, for generality, we express the invariance condition for correlation functions for fields whose positions are restricted, with imaginary time, on a domain D of Euclidean space-time \mathbb{R}^2 , on the boundary of which appropriate boundary conditions are imposed. A true invariance is obtained if g is such that g(D) = D. For $D = \mathbb{R}$, we see that, indeed, the invariances we derived (translations, rotations and scalings) do preserve the domain.

If the symmetry is continuous, we may take it infinitesimal (near the identity), $g_{\epsilon} = \mathrm{id} + \epsilon f$. The infinitesimal transformation of fields is in general of the form $g_{\epsilon}[\mathcal{O}(x)] = \mathcal{O}(x) + \epsilon \Delta \mathcal{O}(x) + \epsilon \Delta \mathcal{O}(x)$ $O(\epsilon^2)$. In this situation, we expect the existence of a local Noether current $j^{\mu}(x)$ that is conserved up to "contact terms":

$$\langle \partial_{\mu} j^{\mu}(x) \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle = -i \sum_j \delta^{(2)}(x - x_j) \langle \mathcal{O}(x_1) \cdots \Delta \mathcal{O}(x_j) \cdots \mathcal{O}(x_n) \rangle$$
(36)

(there may be higher derivatives of the delta function on the right-hand side, but these are not of concern here).

In the previous subsection, we saw how to use locality arguments in order to derive Noether's theorem. Repeating the results: (i) for translation invariance we have $\Delta^{\mu}\mathcal{O}(x) = \partial^{\mu}\mathcal{O}(x)$ and Noether current $j^{\nu} = T^{\mu\nu}$ and therfore $\partial_{\nu}T^{\mu\nu} = 0$ (+ contact terms); (ii) for Lorentz, $\Delta \mathcal{O}(x) = \epsilon_{\mu\rho}x^{\mu}\partial^{\rho}\mathcal{O}(x)$ and $j^{\nu} = \epsilon_{\mu\rho}x^{\mu}T^{\rho\nu}(x)$ whose conservation gives $T^{\mu\nu} = T^{\nu\mu}$ (+ contact terms); (iii) for scale invariance $\Delta \mathcal{O}(x) = x_{\mu}\partial^{\mu}\mathcal{O}(x)$ and $j^{\nu} = x_{\mu}T^{\mu\nu}(x)$ whose conservation implies $T^{\mu}_{\mu} = 0$ (+ contact terms). Therefore, the consequence of these relations is chiral evolution or holomorphicity and anti-holomorphicity, but only up to contact terms.

What can the contact terms look like in general? Here we can go to Euclidean time and use the theory of holomorphic functions. Contact terms must be non-holomorphicity at other fields' positions. Therefore $T(z)\mathcal{O}(x)$ must lead to, inside a correlation function, a function of z that has a Laurent series expansion around x. We combine this insight with the general Wilson's Operator Product Expansion to get

$$T(z)\mathcal{O}(x) = \sum_{n \in \mathbb{Z}} \frac{\mathcal{O}_n(x)}{(z-x)^n}.$$
(37)

It is convenient to define operators on the space of fields L_n which characterize the map from a local field $\mathcal{O}(x)$ at x to the local field $\mathcal{O}_n(x)$ also at x: by translation invariance this map is the same for any x, and we define

$$L_{n-2}\mathcal{O} := \mathcal{O}_n. \tag{38}$$

We similarly have

$$\bar{T}(z)\mathcal{O}(x) = \sum_{n \in \mathbb{Z}} \frac{\bar{\mathcal{O}}_n(x)}{(\bar{z} - \bar{x})^n}$$
(39)

and we define

$$\bar{L}_{n-2}\mathcal{O} := \tilde{\mathcal{O}}_n. \tag{40}$$

In order to specify what local fields some of the $\mathcal{O}_n(x)$ are, we look at the symmetries we have derived. Take $g(z) = \lambda z$ (here z is the complex coordinate in Euclidean space-time). This is a combination of rotation and scaling. Assume the field \mathcal{O} transforms as

$$g[\mathcal{O}(x)] = \lambda^h \bar{\lambda}^h \mathcal{O}(\lambda x). \tag{41}$$

The numbers h, h are called the holomorphic / anti-holomorphic dimensions of \mathcal{O} . In particular, $h + \tilde{h}$ is the scaling dimension d seen for instance in (11) and (12), and $h - \tilde{h}$ can be identified with the spins s of the field. By standard arguments, T has h = 2, $\tilde{h} = 0$. Now we may use the formal relation (which can be verified by integrating on both sides against some arbitrary function)

$$\partial_{\bar{z}} \frac{1}{z} \propto \delta^{(2)}(z). \tag{42}$$

Using this we can identify the $\delta^{(2)}(z)$ terms in $\partial_{\bar{z}}T(z)$, and using the Ward-Takahishi identities we may specify certain terms in the OPE:

$$T(z)\mathcal{O}(x) = \left(\dots + \frac{h}{(z-x)^2} + \frac{\partial_x}{(z-x)} + \dots\right)\mathcal{O}(x).$$
(43)

Here ∂_x is the holomorphic derivative with respect to x. Note that this equation fully specifies the normalization of T(z) (which we had left unspecified until now).

In order to go further, we make the following very important assumption: the set of holomorphic and anti-holomorphic dimensions in the QFT model is bounded from below.

Since, by symmetry, the holomorphic and anti-holomorphic dimensions must be the same on both sides of the OPE (37), and since 1/(z - x) has positive holomorphic dimension, then the holomorphic dimension of \mathcal{O}_n decreases with decreasing n, and thus the series must be truncated from below:

$$T(z)\mathcal{O}(x) = \sum_{n \in \mathbb{Z}, n > n_{\mathcal{O}}} \frac{L_{n-2}\mathcal{O}(x)}{(z-x)^n}.$$
(44)

Looking at \mathcal{O} of lowest possible dimension, as a consequence, there must exist \mathcal{O} (called primaries) such that

$$T(z)\mathcal{O}(x) = \left(\frac{h}{(z-x)^2} + \frac{\partial_x}{(z-x)} + \cdots\right)\mathcal{O}(x).$$
(45)

That is, $L_n \mathcal{O} = 0$ for all $n \ge 1$. In fact, the set of all primaries and descendants under T(z) (higher OPE coefficients of primaries) usually forms a closed OPE algebra, so we may concentrate on these local fields to construct our QFT model.

Equation (45) is a very strong result. Two consequences are as follows.

First, by clustering, we have $\langle T(z)\mathcal{O}(x_1)\cdots\rangle \to 0$ as $z\to\infty$. Then, if all \mathcal{O} 's in the correlation functions are primaries, we may use (45) along with Liouville's theorem in order to get exact expressions for the insertion of a holomorphic stress-energy tensor on the Riemann sphere $\hat{\mathbb{C}}$:

$$\langle T(z)\mathcal{O}_1(x_1)\cdots\rangle = \sum_j \left(\frac{h_j}{(z-x_j)^2} + \frac{\partial_{x_j}}{(z-x_j)}\right) \langle \mathcal{O}_1(x_1)\cdots\rangle.$$
(46)

Second, and perhaps most importantly, there is a larger invariance: local conformal invariance. This may be derived as follows. There exists Noether currents $j_n(z) := z^{n+1}T(z)$ for all $n \in \mathbb{Z}$ such that $\partial_{\bar{z}} j_n(z) = 0 + \text{ contact terms}$. Using the OPE (37) as well as (42), we find that we must have

$$\partial_{\bar{z}} \Big(j_n(z) \mathcal{O}(0) \Big) \propto L_n \mathcal{O}(0) \,\delta^{(2)}(z)$$
(47)

(plus higher derivatives of delta function). These currents are naturally associated to infinitesimal symmetry transformation under $g_n(z) = z + \epsilon z^{n+1}$, and this therefore identifies the corresponding infinitesimal transformation:

$$g_n[\mathcal{O}(0)] = \mathcal{O}(0) + \epsilon L_n \mathcal{O}(0) + \bar{\epsilon} \bar{L}_n \mathcal{O}(0) + O(|\epsilon|^2, \epsilon^2, \bar{\epsilon}^2)$$
(48)

This gives a symmetry meaning to the higher descendants $L_n \mathcal{O}(0)$: these operators appear as a the result of an infinitesimal transformation g_n . In particular, for primary fields and with $n \geq 1$, we have $g_n[\mathcal{O}(0)] = 0$. We may similarly consider transformations of the type $g_{n;x}(z) = z + \epsilon(z-x)^{n+1}$, and we have

$$g_{n;x}[\mathcal{O}(x)] = \mathcal{O}(x) + \epsilon L_n \mathcal{O}(x) + \bar{\epsilon} \bar{L}_n \mathcal{O}(x) + O(|\epsilon|^2)$$
(49)

Expressing an arbitrary g(z) in a series expansion about x, we deduce that the complete transformation of a primary field must be of the form

$$g[\mathcal{O}(x)] = (\partial g(x))^h \, (\bar{\partial}\bar{g}(\bar{x}))^h \mathcal{O}(g(x)). \tag{50}$$

Therefore, we have extended the symmetries (35) to the full groupoid of conformal maps $g: D \to g(D)$. In order to have actual invariance, we need regularity of the currents inside the domain D: invariance occurs by deforming contour integrals of the current up to the boundary of D, and no singularity must be crossed in this process. This requirement is equivalent to asking that the map g be conformal everywhere within D. We note that $j_n(z)$ is singular at ∞ if n > 1, at 0 if n < -1. Hence we have regularity on $\hat{\mathbb{C}}$ if and only if $n \in \{-1, 0, 1\}$, and these generate the *Möbius transformations*. On arbitrary domains D, a true symmetry is only obtain if g(D) = D, and therefore there is invariance under the group of conformal maps that fix D. If g does not fix D, then (35) really only gives a definition of correlation functions on the different g(D).

Finally, let us consider the transformations of the stress-energy tensor itself. We already said that it has h = 2 and $\tilde{h} = 0$. Assume that the lowest dimension is 0, and that the only 0-dimensional fields are multiples of identity $\mathbb{C}1$ (this is often true and sufficient for the derivation, but it is not necessary, and in many models where these don't hold the following conclusions still hold).

Then, using the symmetry $x \leftrightarrow y$ and equality of dimensions on both sides of the OPE, there must exist $c \in \mathbb{C}$ such that

$$T(x)T(y) = \frac{c}{2}\frac{1}{(x-y)^4} + \frac{2T(y)}{(x-y)^2} + \frac{\partial_y T(y)}{(x-y)} + \dots$$
(51)

The quantity c is called the *central charge*, and is a fundamental property of the model of CFT. That is: $L_nT = 0$ for all $n \ge 1$ except for $L_2T = (c/2)\mathbf{1}$. From this, we may obtain g[T(z)] for any infinitesimal $g(z) = z + \epsilon f(z)$ conformal around z = 0. Exponentiating, it turns out that the full transformation is of the form

$$g[T(z)] = \frac{c}{12} \{g, z\} \mathbf{1} + (\partial g(z))^2 T(g(z))$$
(52)

where the Schwarzian derivative is

$$\{g, z\} = \frac{g'''(z)}{g'(z)} - \frac{3}{2} \left(\frac{g''(z)}{g'(z)}\right)^2.$$
(53)

— Exercises —

Exercise 2.1 Using (12) and (41), show that, indeed, we must have d = h + h.

Exercise 2.2 Show that the second relations in (23) and (26) are indeed immediately satisfied by the definitions given of the operators.

Exercise 2.3 Take an arbitrary infinitesimal transformation $g(z) = z + \epsilon h(z)$. Verify that (50) is in agreement, to first order in ϵ , with the transformations (49). Further, verify that (52) is in agreement, to first order in ϵ , with the transformation induced by (51).

Exercise 2.4 Show that, using the $x \leftrightarrow y$ symmetry, there cannot be terms proportional to $(x-y)^{-3}$ in (51).

Exercise 2.5 Equation (38) defines the operators L_n . Further, the OPE (51) defines how to take multiple actions of L_n . Can you derive, from these two equations, the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c\mathbf{l}}{12}\delta_{n+m,0}$$
(54)

satisfied by the operators L_n ?

— Bibliographic notes ——

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3 Basics of integrable QFT

Factorized scattering from local conserved quantities (1h)

This section is an adapted extract from my 2006 / 2008 Oxford / Durham lecture notes on "Introduction to integrable QFT", available on my webpage, see

http://benjamindoyon.weebly.com/lecture-notes.html

3.1 Asymptotic states in massive models

In QFT, the structure of the Hilbert space can be described to a great extent without knowing the details of the Hamiltonian, thanks to relativistic invariance. In general, in local quantum field theory, the Hilbert space is a module for the algebra of space symmetries. A one-particle sub-module is a unitary irreducible module. Here, we have relativistic invariance, and a unitary irreducible module characterized by the real number m (and possibly by other quantum numbers if internal symmetries are present) has basis

$$|\theta, m\rangle \quad (\theta \in \mathbb{R})$$

such that

$$H|\theta,m
angle = m\cosh(\theta)|\theta,m
angle \ , \quad P|\theta,m
angle = m\sinh(\theta)|\theta,m
angle \ , \quad B|\theta,m
angle = -\frac{\partial}{\partial\theta}|\theta,m
angle \ .$$

ລ

This generates an irreducible representation with Casimir

$$H^2 - P^2 = m^2$$

The variable θ , the rapidity, is just a convenient way of parametrizing solutions to this equation which is the usual relativistic dispersion relation. It is natural that we have such a sub-module in \mathcal{H} , because if there is just one stable particle, it cannot interact so it will just propagate freely.

Now, we want quantum field theory to describe the scattering and interaction of particles around a given time and around a given point, and so we assume that if there are many particles, at a finite time before or after the "experimentation", they are separated by a finite distance. Hence, as we go back in time, they whether get more separated, or some may stay together forever, forming bound states. If we see the bound states as particles themselves, at the infinite past, we have many particles infinitely separated, so that they all propagate freely. The associated states, the *in* states, correspond to tensor products of the one-particle states:

$$| heta_1,m_1; heta_2,m_2;\ldots
angle^{in}$$
 .

Certainly the same is true in the infinite future, giving the *out* states:

$$|\theta_1, m_1; \theta_2, m_2; \ldots \rangle^{out}$$
.

Those form two bases for the Hilbert space, with energy

$$H|\theta_1, m_1; \theta_2, m_2; \ldots\rangle^{in,out} = \sum_k m_k \cosh(\theta_k) |\theta_1, m_1; \theta_2, m_2; \ldots\rangle^{in,out}$$

Note that if in the infinite past we have a pure in state, then most likely in the infinite future we will have a linear combinations of *out* states. The overlaps between *in*-states and *out*-states form the scattering matrix, or S-matrix.

Of course, quantum mechanically, a particle cannot have both a definite momentum and a definite position. To be more precise, the construction of such states goes as follows. First, identify a local field $\Psi(x)$ that "creates" the particle in which you are interested. This means that it has the quantum numbers of the particle of interest, and that the Fourier transform of the two-point function

$$\langle \operatorname{vac} | \mathcal{T}(\Psi(x,t)\Psi(0,0)) | \operatorname{vac} \rangle$$

 $(\mathcal{T} \text{ is the time-ordering symbol, bringing the earliest operator to the right – here we assume <math>\Psi$ to be a real field without any charge) has a pole, as function of the square two-momentum $E^2 - p^2$, at the square of the mass m^2 of the particle, and no other singularity at lower values. In superrenormalizable theories, such a field can be uniquely determined by requiring that it be of lowest dimension. This means that correlation functions of such fields will satisfy the Klein-Gordon equation (or other equations depending on the representation of the Lorentz group associated to the particle) asymptotically at large times

$$(m^2 - \partial_x^2 + \partial_t^2) \langle \Psi(x, t) \cdots \rangle = O\left(e^{-m'\sqrt{t^2 - x^2}}\right) \quad \text{as} \quad t^2 - x^2 \to \infty$$

where m' is the lowest mass, greater than m, of a state created by this field, and \cdots represents fields at other fixed positions. Consider the operators (wave packets operators)

$$A(\theta)^{(in,out)} = i \lim_{L \to \infty} \lim_{t \to \pm \infty} \int dx \left(f_{\theta}(x,t) \partial_t \Psi(x,t) - \partial_t f_{\theta}(x,t) \Psi(x,t) \right)$$
(55)

with

$$f_{\theta}(x,t) = \exp\left[\operatorname{i}m \cosh(\theta) t - \operatorname{i}m \sinh(\theta) x - \frac{(x - \coth(\theta)t)^2}{L^2}\right] , \qquad (56)$$

as well as their hermitian conjugate $A^{\dagger}(\theta)^{(in,out)}$. Here we take Ψ to be bosonic with spin 0; simple modifications occur for fermions and for other spins. The operators $A(\theta)^{(in)}$, $A^{\dagger}(\theta)^{(in)}$ satisfy canonical commutation relations, $[A(\theta)^{(in)}, A^{\dagger}(\theta')^{(in)}] = 4\pi\delta(\theta - \theta')$, and similarly for the operators $A(\theta)^{(out)}$, $A^{\dagger}(\theta)^{(out)}$. They are also eigenoperators of the Hamiltonian, $[H, A^{\dagger}(\theta)^{(in,out)}] = m \cosh(\theta) A^{\dagger}(\theta)^{(in,out)}$. Similar operators can be defined for all particles of the theory, and operators corresponding to different particle types commute with each other. The Hilbert space is the Fock space over the algebra of all such *in*-operators, which is isomorphic to the Fock space over the algebra of all out-operators.

In fact, in the wave-packet description above, we chose quite arbitrarily the centers of the wave packets; we have specified them by making all particles "collide at one point" under an extrapolation of their free trajectories. Defining the operators $A(\theta)^{(in,out)}$ by shifting slightly the central positions of the wave packets changes when the particles "would collide", the *impact parameter* – some particles could collide first. That is, we could have used the definition

$$f_{\theta}(x,t) = \exp\left[\operatorname{i}m \cosh(\theta) t - \operatorname{i}m \sinh(\theta) x - \frac{(x - x_0 - \coth(\theta)(t - t_0))^2}{L^2}\right] .$$
(57)

Any choice of $x_0 - \operatorname{coth}(\theta)t_0$ leads to a different but legitimate basis of the Hilbert space, and the overlap between the associated *in*-states and *out*-states form a scattering matrix depending on the impact parameters.

In equations, writing from now on an index a for describing the type of particle (mass, quantum numbers) instead of explicitly writing the mass, the scattering matrix is defined by

$$\begin{aligned} |\theta_1, \theta_2, \dots; \text{imp. param. in} \rangle_{a_1, a_2, \dots}^{in} &= \\ \sum_{a'_1, a'_2, \dots} \int d\theta'_1 d\theta'_2 \cdots S_{a_1, a_2, \dots}^{a'_1, a'_2, \dots} (\theta_1, \theta'_1, \theta_2, \theta'_2, \dots; \text{imp. param. in,out}) \cdot \\ |\theta'_1, \theta'_2, \dots; \text{imp. param. out} \rangle_{a'_1, a'_2, \dots}^{out} \end{aligned}$$

3.2 Elastic scattering

Let us now derive one consequence of the presence of an infinity of local conserved charges in involution in interacting models. Recall that those are local charges that commute with the Hamiltonian and with each other. Besides the Hamiltonian, there is one that is trivial: the momentum. From these two charges, it is convenient to build two different, with better transformation properties under boost, and without scale:

$$Q_1 = m^{-1}(H+P)$$
, $Q_{-1} = m^{-1}(H-P)$

where m is the mass of one of the particles of the theory (arbitrarily chosen, this is just a normalisation of the operator; for instance, take the lowest mass). These charges have eigenvalues

$$Q_{\pm 1}|\theta_1,\theta_2,\ldots\rangle_{a_1,a_2,\ldots}^{in,out} = \sum_k q_{a_k} e^{\pm\theta_k} |\theta_1,\theta_2,\ldots\rangle_{a_1,a_2,\ldots}^{in,out}$$

for $q_{a_k} = m_{a_k}/m$ where m_{a_k} is the mass of the particle of type a_k , and transform with spin ± 1 under boost.

Now consider the set of all local conserved charges $Q = \int dx q(x)$ of the model. It is clear that if Q is a local conserved charge, then so is [B, Q]. Indeed, it is local because [B, q(x)] is a local field, and it is conserved because [[B, Q], H] = [[B, H], Q] + [B, [Q, H]] = [[B, H], Q] = [P, Q] = 0. Thus, we may organize the set of all local conserved charges into representations of the boost operators. These are parametrized by the spin s, so we denote them by Q_s , and we conclude that on any one-particle state, we must have $Q_s |\theta\rangle_a^{in,out} = q_a^{(s)} e^{s\theta} |\theta\rangle_a^{in,out}$.

Remember what locality of a charge $Q = \int dx q(x)$ essentially means: the charge density q(x) commutes with the Hamiltonian density h(x') at different points $x \neq x'$. In order to derive consequences of conserved charges on the scattering matrix, we need something just slightly stronger: the charge density q(x) must be local with respect to a "fundamental" fields creating particles $\Psi(x')$:

$$Q = \int dx q(x) , \quad [q(x), \Psi(x')] = 0 \quad (x \neq x')$$

What we say here, in other words, is that the densities of the conserved charges must be in the consistent set of local observables that include fields that create the particles. Then, a conserved charge applied on an asymptotic state with many particles will act as independently on each of them. That is, it must act as

$$Q_s|\theta_1,\theta_2,\ldots\rangle_{a_1,a_2,\ldots}^{in,out} = \sum_k q_{a_k}^{(s)} e^{s\theta_k} |\theta_1,\theta_2,\ldots\rangle_{a_1,a_2,\ldots}^{in,out} .$$

Note that these charges are *automatically in involution*: the requirement of locality with respect to asymptotic particles guarantees involution.

The numbers $q_{a_k}^{(s)}$ are not determined by these general arguments. Of course, if there is only one type of particle, we can normalise the charges to put all these numbers to 1. But if there are many types, this is not so. However, thanks to the fact that the Q_s commute with any internal symmetry, these numbers will be the same if different particle types fall into a symmetry multiplet. In general, we expect also that they will be different if not. Hence, these numbers tell us about the belonging to symmetry multiplets.

A comment about impact parameters is in order here. We are not writing them explicitly in the states above, but the action of a charge may involve a modification of the impact parameters. We will analyse this in more details below, but for now we just keep in mind that we can choose them as we wich, in such a way that the conditions we derive below are valid for scattering matrices of any impact parameters.

The values of the spins s for which there is a conserved charge and the numbers $q_{a_k}^{(s)}$ are usually good fingerprints of a given integrable model (although there isn't a one-to-one relation). Now, in the equation above, take the *in* states, and sandwich the whole equation with the conjugate *out* state

$$_{a_1',a_2',\cdots}^{out}\langle \theta_1',\theta_2',\ldots|$$

We see that the overlap between an in state and an *out* state is non-zero if and only if the following equation is satisfied:

$$\sum_{k} q_{a_{k}}^{(s)} e^{s\theta_{k}} = \sum_{k} q_{a_{k}'}^{(s)} e^{s\theta_{k}'} \ .$$

This is an infinite set of equations (for all spins s occuring in the model) for an indefinite number of the rapidities θ'_k and particle types a'_k , being given the rapidities θ_k and the particle types a_k . One solution is obviously

$$\{\theta'_k\} = \{\theta_k\} , \quad q_{a'_k}^{(s)} = q_{a_k}^{(s)} \text{ for } \theta'_k = \theta_k .$$

In fact, if we require the solution for the out-going rapidities to be $\{\theta'_k\} = \{\theta_k\}$ for all sets of in-going rapidities $\{\theta_k\}$, then the only possibility for the particle types is $q_{a'_k}^{(s)} = q_{a_k}^{(s)}$ for $\theta'_k = \theta_k$. The meaning of these equalities is that particle types must be the same in the *in* and *out* states, and particles can exchange their rapidities only if they are part of a symmetry multiplets. I am talking of exchanging rapidities, because if particles are part of a symmetry multiplets, conservation of this symmetry implies that particle types *in that symmetry multiplet* must also be the same in the *in* and *out* states. This is elastic scattering: the number of particles going out is the same as that coming in, and they have the same rapidities, with possible exchanges and possible modifications of the particle types.

Remark 3.1 We would indeed like to conclude elastic scattering from the above, but for this we would need to do a little bit more analysis. In principle there may be other solutions than $\{\theta'_k\} = \{\theta_k\}$, depending on how the numbers $q_{a_k}^{(s)}$ behave as function of s. The simplest way to see this is to construct the following analytic function of α , defined by its expansion at large α :

$$f_{a_k}(\alpha) = \sum_{s>0} e^{-s\alpha} q_{a_k}^{(s)}$$

where the sum is over all spins that occur in the model; we assume this series to be convergent for α large enough. Our condition, for positive spins, then becomes

$$\sum_{k} f_{a_k}(\alpha - \theta_k) = \sum_{k} f_{a'_k}(\alpha - \theta'_k) \ \forall \alpha$$

(we can do something similar for negative spins; if we have parity symmetry, it is sufficient to look at positive spins). If the function $f_{a_k}(\alpha)$ has only one singular point on the real line at the same position for all particle types a_k , then the only possibility, for any real $\{\theta_k\}$, is indeed $\{\theta'_k\} = \{\theta_k\}$.

Take, for instance, the situation where there is only one particle type, or when there are many but thay all fall into one multiplet of some internal symmetry. Then, we must have $q_{a_k}^{(s)} = q_{a'_k}^{(s)}$ so that they can all be set to 1. If, for instance, only odd spins are involved (this is the situation in the SU(2)-Thirring model, for instance), the analytical function $f(\alpha)$ is

$$f(\alpha) = \frac{1}{2\sinh(\alpha)}$$

This indeed only has one pole on the real line, and the conclusion follows.

But take, for instance, a fictitious case where we only have odd spins, where for a particle type 1, we have $q_1^{(s)} = 1$, and where for a particle type 2, we have $q_2^{(s)} = 1 + e^{-s\beta}$ for s > 0. Then clearly, if there are just two *in* rapidities and if we have with $\theta_1 = \theta_2 + \beta$ and $a_1 = a_2 = 2$, then we can choose to have only one *out* rapidity with $\theta'_1 = \theta_1$ and $a'_1 = 2$, and the equation is satisfied for all α .

This fictitious situation has never been seen in any model, to my knowledge. In order to establish elastic scattering from this argument, we must check in explicit models what the $q_a^{(s)}$ look like. Admittedly, then, this argument is not as strong as we would like it to be, but generically, we see a singularity only at $\alpha = 0$ in $f(\alpha)$ for $\alpha \in \mathbb{R}$. From this we may conclude that the obvious solution is the only one for real rapidities:

$$\{\theta'_k\} = \{\theta_k\}$$
, $q_{a'_k}^{(s)} = q_{a_k}^{(s)}$ for $\theta'_k = \theta_k$.

3.3 Factorized scattering: Yang-Baxter equations

We consider again the overlap

$$a'_{1,a'_{2,\dots}} \langle \theta'_{1}, \theta'_{2}, \dots; \text{imp. param.} | \theta_{1}, \theta_{2}, \dots; \text{imp. param.} \rangle_{a_{1},a_{2},\dots}$$

and look at the dependence on the impact parameters. It will be convenient and sufficient for our purposes to choose the "same" impact parameters for *in* and *out* particles; that is, for given impact parameters of *in* states, we choose the impact parameters of *out* particles to be those obtained by an extension of the trajectory of the *in* particles all the way to the *out* particles as if they were not interacting. The main point is to write the simple identity:

$${}^{a_1',a_2',\ldots}_{a_1',a_2',\ldots}\langle \theta_1',\theta_2',\ldots; \text{imp. param.}|e^{i\alpha Q_s}e^{-i\alpha Q_s}|\theta_1,\theta_2,\ldots; \text{imp. param.}\rangle^{in}_{a_1,a_2,\ldots} = {}^{a_1',a_2',\ldots}\langle \theta_1',\theta_2',\ldots; \text{imp. param.}|\theta_1,\theta_2,\ldots; \text{imp. param.}\rangle^{in}_{a_1,a_2,\ldots} .$$
(58)

We can exploit this formula by analysing in more details than above the action of a conserved charge on asymptotic states. More precisely, we will see that the scattering matrix is in fact independent of impact parameters, and this will lead to Yang-Baxter equation.

From the formula (55) for the operators destroying/creating asymptotic *in*-states, let us compute

$$[Q_s, A(\theta)^{(in)}] = \lim_{L \to \infty} \lim_{t \to -\infty} \int dx \left(f_\theta(x, t) \partial_t [Q_s, \Psi(x, t)] - \partial_t f_\theta(x, t) [Q_s, \Psi(x, t)] \right)$$

where the function $f_{\theta}(x, t)$ is given in (56). Let us consider for now L finite but large, and take the limit on t. We can evaluate the commutators involved, because the conserved charges act like higher-spin conserved charges on free particles as $t \to -\infty$. For instance, for positive spins we have

$$-\frac{(-\mathrm{i})^{s}q^{(s)}}{m^{s}}\lim_{t\to\infty}\int dx\left(f_{\theta}(x,t)\partial_{t}(\partial_{x}-\partial_{t})^{s}\Psi(x,t)-\partial_{t}f_{\theta}(x,t)(\partial_{x}-\partial_{t})^{s}\Psi(x,t)\right)\,.$$

We can move the derivatives $(\partial_x - \partial_t)$ on the fields to the same derivatives on the wave packet $f_{\theta}(x,t)$ with an overall sign $(-1)^s$. This is obvious for the space derivatives: we do this by integration by part. For the time derivatives, consider the following fact: if a product fg is time independent, then certainly $f\partial_t g = -(\partial_t f)g$. Now for large times, the integral we have is a free-evolving wave packet. We know from quantum mechanics that these evolve in two ways: the centre evolves following a straight trajectory, and the wave packet spreads. The first is taken into account in $f_{\theta}(x,t)$, but the second is not. However, the effect of the spreasing is a term in the exponential in (56) of the form $(x - \coth(\theta)t)^2/L^3\partial L(t)/\partial t$. Now, from $L(t) = \sqrt{L^2 + (t - t_0)^2}$ (with $L(t_0) = L$), we have $\partial L(t) / \partial t \sim 1/(mL)$ for any finite $(t - t_0)$ (that is, of the order of m^{-1}). Since $x - \coth(\theta)t$ is of the order of mL in a wave packet, this correction term is $O((mL)^{-2})$, and we will neglect these terms. Hence, we can move the time derivatives in the limit on t, up to these terms for large L. We must be careful to use time independednce on the difference of the two terms in the definition asymptotic state operators. This allows us to move time derivatives except for terms with only one time derivative, but these don't need to be moved. We then get

$$-\frac{\mathrm{i}^{s}q^{(s)}}{m^{s}}\lim_{t\to-\infty}\int dx\left((\partial_{x}-\partial_{t})^{s}f_{\theta}(x,t)\partial_{t}\Psi(x,t)-\partial_{t}(\partial_{x}-\partial_{t})^{s}f_{\theta}(x,t)\Psi(x,t)\right)\,.$$

Now consider the large L asymptotic: we will keep terms up to in 1/(mL), while always considering $x - \coth(\theta)t$ of the order of L. We have

$$-\frac{\mathrm{i}^{s}q^{(s)}}{m^{s}}(\partial_{x}-\partial_{t})^{s}f_{\theta}(x,t) = -q^{(s)}e^{s\theta}\left[1+\frac{2s}{\mathrm{i}mL\sinh(\theta)}\frac{x-\coth(\theta)t}{L}+O((mL)^{-2})\right]f_{\theta}(x,t) \ .$$

We recognize the leading term as giving the correct eigenvalue of Q_s on asymptotic states, but we want to analyse the effect of the additional term. Let us write it as

$$-\frac{\mathrm{i}^{s}q^{(s)}}{m^{s}}(\partial_{x}-\partial_{t})^{s}f_{\theta}(x,t) = -q^{(s)}\exp\left[s\theta + \frac{2s}{\mathrm{i}mL\sinh(\theta)}\frac{x-\coth(\theta)t}{L} + O((mL)^{-2})\right]f_{\theta}(x,t) \ .$$

In order to calculate

$$e^{i\alpha Q_s}A(\theta)^{(in)}e^{-i\alpha Q_s}$$

we compute

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(-i\alpha \frac{i^s q^{(s)}}{m^s} \right)^n (\partial_x - \partial_t)^{ns} f_\theta(x, t)$$

= $\exp\left(-i\alpha q^{(s)} \exp\left[s\theta + \frac{2s}{imL\sinh(\theta)} \frac{x - \coth(\theta)t}{L} + O((mL)^{-2}) \right] \right) f_\theta(x, t)$

Expanding at large L, the right-hand side can be written

$$\exp\left(-\mathrm{i}\alpha q^{(s)}e^{s\theta}\right)\exp\left[\mathrm{i}M\cosh(\theta)t - \mathrm{i}M\sinh(\theta)x - \frac{(x - x_0 - \coth(\theta)(t - t_0))^2}{L^2} + O((mL)^{-2})\right]$$

with

$$m \cosh(\theta) t_0 - m \sinh(\theta) x_0 = \alpha q^{(s)} s e^{s\theta}$$
.

It is convenient to choose have $x_0 = -t_0 \equiv l$.

We recognize this as, up to the factor $\exp\left(-i\alpha q^{(s)}e^{s\theta}\right)$ containing the eigenvalue of Q_s , simply the wavepacket (57) with non-zero impact parameters. The impact parameter is given by

$$ml = -\alpha q^{(s)} s e^{(s-1)\theta} \quad (s>0) \; .$$

If s = 1, it is clearly independent of θ , but otherwise, the impact parameters depend on the rapidity θ .

A similar derivation holds for negative spins. The action of Q_s on a field $\Psi(x,t)$ in this case is

$$[Q_s, \Psi(x, t)] = -\frac{(-i)^s q^{(s)}}{m^{|s|}} (\partial_x + \partial_t)^{|s|} \Psi)(x, t) \quad (s < 0)$$

and we will have $x_0 = t_0 \equiv l$. The impact parameter then takes the value

$$ml = -\alpha q^{(s)} s e^{(s+1)\theta} \quad (s < 0) .$$

Hence, we found that

$$e^{i\alpha Q_s} A(\theta)^{(in)} e^{-i\alpha Q_s} = e^{-i\alpha q^{(s)} e^{s\theta}} \left. A(\theta)^{(in)} \right|_{\text{impact parameter } l}$$

and the same thing holds, for conjugate eigenvalue, for the operator $A^{\dagger}(\theta)^{(in)}$. Looking back at (58), we realize that the scattering matrix is invariant under such changes of impact parameters.

It is not a priori clear that using enough conserved charges, we can bring the impact parameters to any value we want, hence showing that the scattering matrix is independent of impact parameters. However, there is a simple and strong argument that can do this. First, we must realise that the 2-particle S-matrix does not depend on the impact parameters. Indeed, by convention, we chose the *out* impact parameters to agree with the *in* impact parameters (which can always be done in elastic scattering). Then, space and time translation invariance means that the 2-particle S matrix is independent of impact parameters. Consider then the 3-particle to 3-particle scattering, with scattering matrix

$$S^{a'_1,a'_2,a'_3}_{a_1,a_2,a_3}(\theta_1,\theta_2,\theta_3;l_1,l_2,l_3)$$
.

Here we use elastic scattering and write only the three rapidities involved. The particle types a_1, a_2, a_3 are for the *in*-particles of rapidity $\theta_1, \theta_2, \theta_3$ respectively, and similarly for a'_1, a'_2, a'_3 for the *out*-particles. The parameters l_1, l_2, l_3 are the associated impact parameters. Consider the set of rapidities $(\theta_1, \theta_2, \theta_3)$ for which the three impact parameters resulting from the action of Q_s ,

$$-\alpha m_{a_i}^{-1} q_{a_i}^{(s)} s e^{(s-1)\theta}$$
, $i = 1, 2, 3$

are not all equal to each other. There is only possibly one set $(\theta_1, \theta_2, \theta_3)$ for which this is not so for a given spin s; for this one, we must use another conserved charge of a different spin and verify that the inequality holds. If we have parity invariance in the theory, we can use the conserved charge of spin -s (we have $q_a^{(-s)} = q_a^{(s)}$ in parity invariant models), and we are guaranteed that it will work. Then, we can use invariance under shift of impact parameters for each particle, and with α large enough, the impact parameters become so different that the particle will only meet pair by pair at points very distant from each other: say particles 1 and 2 first, then particle particles 1 and 3, finally particles 2 and 3. Since these meeting points are very far from each other, by locality the 3-particle S-matrix decomposes itself into products of 2-particle S-matrices:

$$S_{a_1,a_2,a_3}^{a_1',a_2',a_3'}(\theta_1,\theta_2,\theta_3;l_1,l_2,l_3) = S_{a_1,a_2}^{c,b}(\theta_1,\theta_2)S_{c,a_3}^{a_1',d}(\theta_1,\theta_3)S_{b,d}^{a_3',a_2'}(\theta_2,\theta_3)$$

(with implicit sum over repeated indices). Since this does not depend anymore on the initial impact parameters, this shows that the scattering matrix does not depend on them. Hence,

we can also choose them so that we get the opposite situation: particles 2 and 3 meet first, then particles 1 and 3, finally particles 1 and 2 (equivalently, we could have taken α large with opposite sign). The two ways of decomposing it give the same value:

$$S_{a_1,a_2}^{c,b}(\theta_1,\theta_2)S_{c,a_3}^{a_1',d}(\theta_1,\theta_3)S_{b,d}^{a_2',a_3'}(\theta_2,\theta_3) = S_{a_2,a_3}^{b,c}(\theta_2,\theta_3)S_{a_1,c}^{d,a_3'}(\theta_1,\theta_3)S_{d,b}^{a_1',a_2'}(\theta_1,\theta_2)$$
(59)

This equation (or set of equations) is called *Yang-Baxter equation*.

Now consider the multi-particle scattering. We can apply a similar argument and write it as a product of a n-1-particle S-matrix times a 2-particle S-matrix. Recursively, we then get a product of 2-particle S-matrices. Again this is true for any initial impact parameters, so that it does not depend on them. Repeated use of Yang-Baxter equation insures that we can write it as any decomposition into 2-particle S-matrices.

Note that we only needed *two* conserved charges of spin higher than 1 to do these manipulations! Using similar arguments, Parke (1980) was able to actually prove *elastic scattering* as well as factorizability, using two local conserved charges of higher spin.

It is worth noting that similar arguments could be attempted in higher dimension. The presence of higher-spin conserved charges indeed give there the independence form impact parameters, and in higher than on space dimension, this means that we can choose the trajectory to all avoid each other. Hence the theory has to be trivial: free fermions or free bosons. This is the essence of Coleman-Mandula theorm (1967).

3.4 Analytic and other properties of the two-particle S-matrix

Besides satisfying the Yang-Baxter equation (59), the two-particle S-matrix must satisfy other, simpler equations that are consequences of general properties of QFT, as well as "bootstrap" equations, which are like Yang-Baxter equations but which concern bound states.

The first equation comes from Lorentz invariance, and says that only the difference of rapidities matters:

$$S_{a_1,a_2}^{b_1,b_2}(\theta_1,\theta_2) = S_{a_1,a_2}^{b_1,b_2}(\theta_1-\theta_2)$$
.

The second is about the fact that the transformation from *in*-states to *out*-states is unitary. From $|\theta_1, \theta_2\rangle_{a_1, a_2}^{(in)} = \sum_{b_1, b_2} S_{a_1, a_2}^{b_1, b_2} (\theta_1 - \theta_2) |\theta_1, \theta_2\rangle_{b_1, b_2}^{(out)}$ and orthonormality of both *in*-states and *out*-states, we have

$$\sum_{b_1,b_2} S^{b_1,b_2}_{a_1,a_2}(\theta) \left(S^{b_1,b_2}_{c_1,c_2}(\theta) \right)^* = \delta^{c_1}_{a_1} \delta^{c_2}_{a_2} \tag{60}$$

(for real θ). Note that the unitarity relation for non-integrable models is not this, because there is in general more intermediate states, with more than 2 particles.

With time-independent Hermitian Hamiltonian, there is time-reversal symmetry. This says that if we reverse the direction of time and if all particles are replaced by their corresponding anti-particle, then the scattering amplitudes are the same. The consequence for the two-particle scattering matrix is

(time-reversal symmetry)
$$S^{b_1,b_2}_{a_1,a_2}(\theta) = S^{\bar{a}_2,\bar{a}_1}_{\bar{b}_2,\bar{b}_1}(\theta)$$

(for real rapidities). The exchange of horizontal position of the indices occurs because if a particle is travelling to the right, then under time reversal, its anti-particle travels to the left. The notation \bar{a} means that we must take the particle index corresponding to the anti-particle of that of particle type a. Generically, there will be a conjugation matrix $C_{a,b}$ such that

$$A^{\bar{a}} = C_{a,b} A^b , \quad A_{\bar{a}} = C^{a,b} A_b ,$$

with property

$$C_{a,b}C^{b,c} = \delta_a^c \; .$$

The most important properties of the scattering matrix is its analytical structure. General principles of QFT (or of the theory of analytic S-matrices) say the following:

In a general model of (1+1-dimensional) QFT, as an analytic function of the Mandelstam variable

$$s = m_1^2 + m_2^2 + 2m_1m_2\cosh(\theta_1 - \theta_2)$$
,

the two-particle to two-particle scattering matrix (preserving the masses – such a scattering is always elastic from the kinematics) is a multi-valued function, and possesses a Riemann sheet with only three branch points where it is otherwise meromorphic. On this sheet, called the physical sheet, the branch points are at $(m_1 + m_2)^2$, $(m_1 - m_2)^2$ and ∞ , the cuts are on the real line avoiding the interval $[(m_1 - m_2)^2, (m_1 + m_2)^2]$, and the only possible poles are on this interval. This sheet is characterised by the fact that the physical values of the scattering matrix are just above the cut on the interval $[(m_1 + m_2)^2, \infty]$ of the real line.

The prescription "just above the cut" for the physical values of the scattering can be understood as coming from Feynmann's prescription for the propagator. The poles on the real line, between the cuts, correspond to possible bound states between the two particles involved in the scattering. The restriction to 1+1-dimensional QFT implies that the 2-particle scattering matrix only depends on the Mandelstam variable s.

Back in the $\theta = \theta_1 - \theta_2$ plane, the physical sheet corresponds to the strip $\text{Im}(\theta) \in [0, \pi]$, the branch points are $\theta = 0, i\pi, \infty$, and the cuts run along the lines $\text{Im}(\theta) = 0, \pi$ connecting to infinity. They may be chosen to run towards the left or towards the right. The poles corresponding to bound states lie on the line $\text{Re}(\theta) = 0$ in the physical strip.

An important relation valid on the physical sheet is called "Hermitian analyticity", or when there is parity invariance, "real analyticity". It simply says that the complex conjugate of the scattering matrix on the physical sheet is the scattering matrix on the complex conjugate argument still on the physical sheet:

$$\left(S_{a_1,a_2}^{b_1,b_2}(\theta)\right)^* = S_{b_2,b_1}^{a_2,a_1}(-\theta^*) \ . \tag{61}$$

Notice how the real part of the rapidity changes its sign. This is because $s \mapsto s^*$ on the physical sheet corresponds to $\theta \mapsto -\theta^*$.

Remark 3.2 Hermitian analyticity can be understood as follows. Consider the path-integral formulation of the scattering matrix, with real rapidities and $\theta_1 > \theta_2$:

$$S^{b_1,b_2}_{a_1,a_2}(\theta) = \int_{\substack{\Psi_{a_k} \propto A(x-c_k)e^{iE_kt-ip_kx} \ (c_1 < c_2, \ t \to -\infty) \\ \Psi_{b_k} \propto A(x-c'_k)e^{iE_kt-ip_kx} \ (c'_1 > c'_2, \ t \to +\infty)}} [d\Psi] e^{iS[\Psi]} \ .$$

Here, the Ψ_k are the well-separated wave packets, and A is a Gaussian envelope, as before. The numbers c_k , c'_k are the centers of the envelopes. Taking the complex conjugate of this gives

$$\left(S^{b_1,b_2}_{a_1,a_2}(\theta) \right)^* = \int_{\substack{\Psi_{a_k} \propto A(x-c_k)e^{-iE_kt+ip_kx} \ (c_1 < c_2, \ t \to -\infty) \\ \Psi_{b_k} \propto A(x-c'_k)e^{-iE_kt+ip_kx} \ (c'_1 > c'_2, \ t \to +\infty) }} [d\Psi] e^{-iS[\Psi]} \ .$$

On the other hand, time-reversal invariance can be written

$$S_{a_{1},a_{2}}^{b_{1},b_{2}}(\theta) = \int_{\substack{\Psi_{a_{k}} \propto A(x-c_{k})e^{-iE_{k}t-ip_{k}x} (c_{1} < c_{2}, t \to +\infty) \\ \Psi_{b_{k}} \propto A(x-c_{k}')e^{-iE_{k}t-ip_{k}x} (c_{1}' > c_{2}', t \to -\infty)}} [d\Psi] e^{-iS[\Psi]} \quad (\theta_{1} > \theta_{2})$$

Comparing gives (62) for θ real and positive, if we consider that the region $\theta_1 < \theta_2$ of the latter expression is obtained by analytic continuation in θ . Analytic continuation gives it for all θ on the physical sheet.

If there is parity invariance:

$$(\text{parity invariance}) \qquad S^{b_1,b_2}_{a_1,a_2}(\theta) = S^{b_2,b_1}_{a_2,a_1}(\theta)$$

then Hermitian analyticity implies real analyticity,

$$\left(S_{a_1,a_2}^{b_1,b_2}(\theta)\right)^* = S_{a_1,a_2}^{b_1,b_2}(-\theta^*) , \qquad (62)$$

because we always have CPT invariance (charge-parity-time-reversal invariance) from general principles of QFT:

(CPT invariance)
$$S^{b_1,b_2}_{a_1,a_2}(\theta) = S^{a_1,a_2}_{b_1,b_2}(\theta)$$
.

Now we may combine Hermitian analyticity with the unitarity relation of integrable models derived above, in order to obtain

$$S_{a_1,a_2}^{b_1,b_2}(\theta)S_{b_2,b_1}^{c_2,c_1}(-\theta^*) = \delta_{a_1}^{c_1}\delta_{a_2}^{c_2} .$$
(63)

This relation is usually called "unitarity" for the 2-particle S-matrix in integrable models. It is important to recall that $-\theta^*$ means the analytic continuation from θ two $-\theta^*$ counterclockwise around the point 0 (that is, for physical initial $\theta > 0$, we always stay on the physical strip to reach $-\theta$).

Another relation, again consequence of general principles of QFT, is crossing symmetry. It essentially says that quantizing the theory in a scheme where the time and space arrows are rotated by $\pi/2$ gives the same scattering amplitudes. For the 2-particle scattering $S_{a_1,a_2}^{b_1,b_2}(\theta_1-\theta_2)$, the slope of the world line of particle k is $\operatorname{coth}(\theta_k)$. Making a $\pi/2$ counter-clockwise rotation amounts to $\theta_k \mapsto i\pi/2 - \theta_k$, to be understood as analytic continuation on the physical strip, and after this rotation particle 2 seems like propagating in reverse time. We can then use time-reversal symmetry in order to transform it to an anti-particle propagating correctly, and we must remember that when complex values of rapidities are involved, time-reversal symmetry involves complex conjugation of the rapidities (understood, again, on the physical strip of $\theta_1 - \theta_2$). Hence, we find

$$S_{a_1,a_2}^{b_1,b_2}(i\pi - \theta) = S_{\bar{b}_2,a_1}^{\bar{a}_2,b_1}(\theta)$$
(64)

Now it is possible to understand the nature of the branch points at $\theta = 0, i\pi$ on the physical strip. Start at $\theta > 0$ real, and use the unitarity relation (63) in order to travel to $\theta < 0$ real, always staying on the physical strip. Using again (63) with $\theta < 0$ real, we get the analytic continuation all the way around the point 0. But this is equal to the initial value:

$$S_{a_1,a_2}^{b_1,b_2}(\theta)S_{b_2,b_1}^{c_2,c_1}(-\theta^*)[S_{c_1,c_2}^{d_1,d_2}(\theta)]_{\text{anal. cont. around }0} = S_{a_1,a_2}^{d_1,d_2}(\theta) = [S_{a_1,a_2}^{d_1,d_2}(\theta)]_{\text{anal. cont. around }0} = [S_{a_1,a_2}^{d_1,d_2}(\theta)]_{\text{anal. cont. cont. around }0} = [S_{a_1,a_2}^{d_1,d_2}(\theta)]_{\text{anal.$$

(we used unitarity in two different ways for contracting the two different pairs of S-matrices). Hence, the point $\theta = 0$ is in fact not a branch point, it is a regular point without any singularity; this means that the corresponding branch point in the s-plane is a square-root branch point. By crossing symmetry, the same is true for the point $\theta = i\pi$.

This means that the S-matrix is a meromorphic function of θ , its poles on the physical strip can only be on the imaginary axis, and once it is determined on the physical sheet, it is determined everywhere on the θ -plane. Also, if the poles and their residues are known on the physical strip and on the strip Im $(\theta) \in [-\pi, 0]$, then it is determined everywhere.

Remark 3.3 There are additional conditions coming from possible bound states. A bound state manifests itself by a virtual particle being created in a two-particle scattering process, and gives rise to a pole in the S-matrix at the corresponding imaginary value of the rapidity. This value is sole consequence of the kinematics. For two particles of masses m_1 and m_2 forming a particle of mass m, it is

$$\cosh(\theta) = \frac{m^2 - m_1^2 - m_2^2}{2m_1 m_2} .$$
(65)

The residue at the pole is purely imaginary, iR. If R > 0, this is a bound state in the "direct" channel, if R < 0, it is in the "crossed" channel (as if the virtual particle were travelling faster than the speed of light!). This would not be of much use if it were not for the resulting "bound-state Yang-Baxter" relations. They are not as easy to derive from local conserved charges, and I will not go into any detail of there derivation; but they are easy to state. They can just be seen as consistency in the different ways of decomposing a 3-particle scattering, where two of the particles (say particle 1 and 2) form a virtual bound-state particle, into products of 2-particle scattering. The assumption is that any bound state is a particle that is part of the asymptotic state spectrum (this assumption is called "nuclear democracy"), hence particle 3 can scatter with the bound-state of particles 1 and 2 as if it were an asymptotic particle, and this must give the same result as when particle 3 scatters with particles 1 then 2 separately.

3.5 Simple examples

3.5.1 Recapitulation of the requirements

The problem of finding the scattering matrix is now reduced to the problem of solving a set of equations along with analytical conditions (a Riemann-Hilbert problem). We want to find a function $S_{a_1,a_2}^{b_1,b_2}(\theta)$ that is meromorphic and that satisfies:

- 1. Yang-Baxter equation (59);
- 2. Hermitian analyticity (62);
- 3. unitarity (63);
- 4. crossing symmetry (64);
- 5. bound state consistency relations when poles on the physical strip (65) are present.

The solutions to this problem can be characterized by the possible structure of solutions to the Yang-Baxter equations.

3.5.2 A diagonal example

The simplest structure is diagonal scattering: $S_{a_1,a_2}^{b_1,b_2}(\theta) = \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} S_{a_1,a_2}(\theta)$ (no sum over repeated indices!). This trivially solves Yang-Baxter relations, and we are left only with the other requirements. The most general solution to requirements of points 2, 3 and 4 above is the following:

$$S_{a,b}(\theta) = \prod_{x \in X_{a,b}} \frac{\sinh\left(\frac{1}{2}(\theta + i\pi x)\right)}{\sinh\left(\frac{1}{2}(\theta - i\pi x)\right)}$$
(66)

where $X_{a,b}$ is a set of a finite number of complex numbers lying, for instance, in the strip $\operatorname{Re}(x) \in [-1,1]$. If these numbers are all in the strip $\operatorname{Re}(x) \in [-1,0]$, then there are no poles on the physical strip, and this immediately is a valid scattering matrix! An example of such a model is the so-called sinh-Gordon model. It is a model with only one particle in the spectrum

of asymptotic states and, besides the mass of this particle, one additional free real parameter b. The scattering amplitude is

$$S(\theta) = \frac{\tanh\left(\frac{\theta}{2} - \frac{\mathrm{i}\pi b^2}{2(b^2+1)}\right)}{\tanh\left(\frac{\theta}{2} + \frac{\mathrm{i}\pi b^2}{2(b^2+1)}\right)}$$

The classical action of the sinh-Gordon model is written in terms of a single real blosonic field ϕ :

$$\mathcal{A} = \int \mathrm{d}x \,\mathrm{d}t \,\left\{ \frac{1}{16\pi} \left[(\partial_t \phi)^2 - (\partial_x \phi)^2 \right] - 2\mu \cosh(b\phi) \right\} \;.$$

The mass of the particle is proportional to a power of μ . The fact that the scattering amplitude above corresponds to this action is really a conjecture, but I will discuss below possible ways of going from classical action to a solution to the equations for the scattering matrix.

If poles are present on the physical strip, we must additionally solve the consistency relations for bound states. I will not go at all into this quite extensive subject...

3.5.3 A non-diagonal example

Another class of solutions to the Yang-Baxter equations are those where two types particles (that we will characterize by two charges +/-), scatter into one another, with scattering matrix of the form

$$S(\theta) = \begin{pmatrix} a(\theta) & 0 & 0 & 0 \\ 0 & b(\theta) & c(\theta) & 0 \\ 0 & c(\theta) & b(\theta) & 0 \\ 0 & 0 & 0 & a(\theta) \end{pmatrix} \stackrel{++}{-+}$$

The pairs of signs on the right are the lower indices, and those on the top are the upper indices of the scattering matrix $S_{a_1,a_2}^{b_1,b_2}(\theta)$. The Yang-Baxter equation then amounts to only two equations:

$$a(\theta_{12})b(\theta_{13})c(\theta_{23}) = b(\theta_{23})c(\theta_{12})c(\theta_{13}) + a(\theta_{13})b(\theta_{12})c(\theta_{23})$$

$$a(\theta_{12})a(\theta_{23})c(\theta_{13}) = a(\theta_{13})c(\theta_{12})c(\theta_{23}) + b(\theta_{12})b(\theta_{23})c(\theta_{13})$$
(67)

where $\theta_{ij} \equiv \theta_i - \theta_j$. From this, one then needs to solve the four other requirements written above. A solution that has no pole on the physical strip is that of the so-called SU(2)-Thirring model. It is a model with two particles of equal mass and of opposite SU(2) spin, transforming under the fundamental representation of SU(2), without any other free parameter than the mass and having a scattering matrix as above with

$$a(\theta) = \frac{\Gamma\left(\frac{1}{2} - \frac{\mathrm{i}\theta}{2\pi}\right)\Gamma\left(\frac{\mathrm{i}\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{2} + \frac{\mathrm{i}\theta}{2\pi}\right)\Gamma\left(-\frac{\mathrm{i}\theta}{2\pi}\right)} , \quad b(\theta) = a(\theta)\frac{\theta}{\mathrm{i}\pi - \theta} , \quad c(\theta) = a(\theta)\frac{\mathrm{i}\pi}{\mathrm{i}\pi - \theta} . \tag{68}$$

The classical action of this model is written in terms of a Dirac spinor Ψ with extra index in the fundamental representation of SU(2):

$$\mathcal{A} = \int \mathrm{d}x \,\mathrm{d}t \,\left(\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi - \frac{g}{2}\sum_{m}\bar{\Psi}\gamma_{\mu}\sigma^{m}\Psi\bar{\Psi}\gamma^{\mu}\sigma^{m}\Psi\right)$$

where σ^m are Pauli matrices and γ^{μ} are Dirac gamma-matrices. The parameter g, which is classically dimensionless, is in fact a running coupling constant under the action of the renormalisation group, and the scale associated to this running gives rise to the physical mass of the particle; this process is called "dimensional transmutation".

3.6 Additional remarks

Remark 3.4 Zamolodchikov-Faddeev algebra. A convenient way of representing the properties of factorized scattering is using the Zamolodchikov algebra. It is the exchange algebra generated by elements $Z_a(\theta)$, which are parametrised by particle type *a* and rapidity θ . This algebra is used to represent states in integrable QFT by associating to a state an element of the enveloping algebra of Zamolodchikov's algebra. The horizontal position of each factor corresponds to the position of the particles in the scattering process at a given time. For instance, the *in*-state, where wave packets are ordered from the most positive rapidity at the left to the most negative one at the right, is represented by the product (an element of the enveloping algebra)

$$Z_{a_1}(\theta_1)\cdots Z_{a_n}(\theta_n)$$
, $\theta_1 > \cdots > \theta_n$.

Since, as we saw, we can choose the impact parameters so that scattering occurs 2 particles by 2 particles at points very separated from each other, we can define "intermediate states", which are neither *in* nor *out*, where wave packets travel freely and are ordered in such a way that some may never meet in the future (going in opposite directions after having scattered already). Such states would not make sense in ordinary quantum field theory, or at least would be of no interest, because in this case such a situation never happens by construction (the asymptotic states are initially defined such that particles meet in a finite region of space-time); it only happens, or is useful, thanks to independence from impact parameters. Then, in general, such intermediate states will be represented by

$$Z_{a_1}(\theta_1) \cdots Z_{a_n}(\theta_n)$$
, any ordering of $\theta_1, \ldots, \theta_n$

Physical scattering tells us that the Zamolodchikov algebra elements must satisfy the exchange relation

$$Z_{a_1}(\theta_1)Z_{a_2}(\theta_2) = S_{a_1,a_2}^{b_1,b_2}(\theta_1 - \theta_2)Z_{b_2}(\theta_2)Z_{b_1}(\theta_1) , \quad \theta_1 > \theta_2 .$$
(69)

In order to verify associativity of this algebra, it is sufficient to check that the two ways of obtaining $Z_{a_3}(\theta_3)Z_{a_2}(\theta_2)Z_{a_1}(\theta_1)$ from $Z_{a_1}(\theta_1)Z_{a_2}(\theta_2)Z_{a_3}(\theta_3)$ are consistent. They are indeed consistent thanks to the Yang-Baxter equation (59).

The relation (69) was written for $\theta_1 > \theta_2$. The exchange relation for $\theta_1 < \theta_2$ follows from it by exchanging θ_1 and θ_2 , but if we want to write it in the same form as (69), we need to define $S_{a_1,a_2}^{b_1,b_2}(\theta_1 - \theta_2)$ for $\theta_1 < \theta_2$. The algebra tells us that it should be defined such that the following equation is sastisfied:

$$S_{a_1,a_2}^{b_1,b_2}(\theta_1-\theta_2)S_{b_2,b_1}^{c_2,c_1}(\theta_2-\theta_1) = \delta_{a_1}^{c_1}\delta_{a_2}^{c_2}$$

This is just a consequence of the algebra, but we may wonder if the analytic continuation of the physical 2-particle scattering matrix $S_{a_1,a_2}^{b_1,b_2}(\theta_1 - \theta_2)$ from the region $\theta_1 > \theta_2$ to the region $\theta_1 < \theta_2$ would give a function that satisfies this relation. It is not a priori clear that this must be so (and certainly it is not true of a 2-particle scattering matrix in non-integrable models), but since the states we constructed are actual asymptotic-like state of integrable models, we could expect that the relation above is true for the 2-particle scattering matrix as an analytical function of the rapidities. Comparison with the previous sub-section shows that it is so.

It is just a small step from representing states using the enveloping algebra of Zamolodchikov's algebra to constructing the Hilbert space as a module for a slightly extended algebra, Zamolodchikov-Faddeev algebra. It is defined by the relations

$$Z_{a_1}(\theta_1)Z_{a_2}(\theta_2) - S_{a_1,a_2}^{b_1,b_2}(\theta_1 - \theta_2)Z_{b_2}(\theta_2)Z_{b_1}(\theta_1) = 0$$

$$\bar{Z}^{a_1}(\theta_1)\bar{Z}^{a_2}(\theta_2) - S_{b_1,b_2}^{a_1,a_2}(\theta_1 - \theta_2)\bar{Z}^{b_2}(\theta_2)\bar{Z}^{b_1}(\theta_1) = 0$$

$$Z_{a_1}(\theta_1)\bar{Z}^{a_2}(\theta_2) - S_{b_2,a_1}^{a_2,b_1}(\theta_2 - \theta_1)\bar{Z}^{b_2}(\theta_2)Z_{b_1}(\theta_1) = 2\pi\delta_{a_1}^{a_2}\delta(\theta_1 - \theta_2) .$$
(70)

The Hilbert space can then be constructed as a Fock space over this algebra, with vacuum defined by $\bar{Z}^a(\theta)|\text{vac}\rangle = 0$, and with conjugation defined by $Z_a(\theta)^{\dagger} = \bar{Z}^a(\theta)$. Consistency of these relations is a consequence of Yang-Baxter relation (59) and unitarity (63), and consistency with the Hermitian structure is consequence of Hermitian analyticity (62). **Remark 3.5** *CDD ambiguity.* It is important to note that in general, any solution to the 5 requirements above can always be multiplied by a factor (66) for $\operatorname{Re}(x) \in [-1, 0]$, giving another solution. This is called a CDD factor (from Castillejo, Dalitz and Dyson, 1956), and the resulting ambiguity for the scattering matrix is the CDD ambiguity. The sinh-Gordon example is a pure CDD factor⁴. Resolving the CDD ambiguity cannot be done solely using the techniques of factorized scattering theory; one needs to do perturbative calculations of the scattering matrix or other, non-perturbative checks.

Remark 3.6 From classical action to scattering matrix. We have seen some examples of solutions to the set of requirements for the scattering matrix. We also related these examples to classical actions. How to figure out such a relation? This is in general a very complicated problem for which there is no general solution. However, we can often go a long way towards its solution in certain situations.

1. Sometimes it is possible to determine the spectrum of asymptotic particles (masses and types) from semi-classical calculations on the classical action. This occurs when we can write down explicitly classical solutions to the equation of motion. Every asymptotic state corresponds to a given solution to the classical equation of motion. Indeed, suppose we have an action depending on a field ϕ (hence an equation of motion for this field). Then we can formulate the matrix elements of operators between two given asymptotic states via a path integral with asymptotic conditions on ϕ at $t = \pm \infty$ corresponding to the large-time limit of appropriate classical solutions⁵. The most pictorial situation is when the classical solutions are soliton-like. Then, we understand that the asymptotic particles are actually just quantization of the solitons (bumps or kinks that seem to propagate locally in classical solutions without being affected by one another at large distances)⁶.

If both in and out asymptotic states are the same, then, in order to evaluate approximately these matrix elements, we can just make a saddle-point expansion of the integrand around the required classical solution; this is the WKB approximation scheme. Of course this is an approximate method in general, but the magic occurs when we want to calculate the masses of the particles and of the bound states that they can form using WKB approximation: it turns out that the results to leading order are exact in integrable models! There is no proof of that, it is more an "empirical" result, but there is an understanding as follows. Recall that in integrable models of classical mechanics (where everything is clear), the orbit on phase space always falls onto tori. But we know that for systems of this type, the "old quantization method" from Bohr, by which we simply quantize periodic orbits by requiring that the total phase space area be a multiple of \hbar , is actually correct: take the example of the hydrogen atom (which is an integrable model). This old quantization method is essentially the leading order of the WKB approximation. Transferring all this to integrable QFT (neglecting the details about the infinity of degrees of freedom!) we may understand the result above. Once we have the masses of the particles and the bound states they can form, there is little ambiguity in the solution to the requirements above that we must take, except mainly for the CDD ambiguity. To fix all ambiguities, one generecally needs to perform some perturbative calculations or checks of other types.

2. Another way of making quite explicit calculations from the classical action is by applying the methods of "coordinate Bethe ansatz." This aims at solving the system explicitly by constructing the Hilbert space starting from a simple reference state ("pseudo-vacuum") and by making a simple ansatz for the excited states. After lengthy and tedious calculation this sometimes leads to the full spectrum and even to the scattering matrix itself.

3. Finally, it is also sometimes possible to have an explicit, integrable "lattice regularisation" of the QFT, or quantum-chain regularisation. Integrable quantum chains and lattice models can be solved by the method of algebraic Bethe ansatz, and, although from this solution it is still very hard to go back

 $^{^{4}}$ Hence, its scattering amplitude is just a CDD factor times the scattering amplitude of a free massive Majorana fermion!

 $^{^{5}}$ To make a connection with the previous description of asymptotic states via a "fundamental fields" creating particles, a fundamental field for some particles is essentially a field such that its large-time asymptotic in some classical solutions looks, in a sense that can be made precise, like separated free plane waves.

⁶Again in connection to the concept of "fundamental field", the field, say ϕ , whose classical configurations are solitons is not the "fundamental field" associated to the particles corresponding to its solitons. Indeed, solitons do not look like plane waves at large time! Generically, the fundamental field for such particles is a very complicated functional of ϕ .

to the QFT model and often not known how to do that, we may still have enough intuition about the spectrum in order to determine the scattering matrix.

Remark 3.7 The "scattering matrix" formulation of QFT. Recall how I defined a QFT: a Hilbert space with a local Hamiltonian bounded from below and local space-time symmetry generators. Now that we have an idea about how to go from a given Hamiltonian to the scattering matrix, it is worth noting that a local massive QFT can equivalently be defined by giving a set particles and masses (hence having the Hilbert space and the action of the Hamiltonian on it) and a scattering matrix with specific properties. The specific properties are complicated in general (this is the theory of the analytic S-matrix), but in integrable models, they are just as described above. Note that just giving a set of particles and masses, although it directly gives the Hilbert space and the action of the Scattering matrix. This is quite non-trivial; for instance, how do we define local fields if we don't have an explicit Hamiltonian density? How do we calculate their correlation functions? These questions are what is assessed in the form factor program of integrable QFT.

—— Exercises ——

Exercise 3.1 An integrable model has an infinity of local conserved charges. It is very instructive to see what they look like in a simple example. The most simple example possible of conserved charges in an integrable model are those of the free boson. The Hamiltonian of the free boson is

$$H = \int dx \, \frac{1}{2} \left[\pi^2(x) + \left(\frac{\partial}{\partial x} \phi(x) \right)^2 + m^2 \phi^2(x) \right]$$

with the two local fields $\phi(x)$ and $\pi(x)$ satisfying

$$[\phi(x),\pi(x')] = i\delta(x-x') \ .$$

The momentum operator is simply expressed as

$$P = -\int dx \,\pi(x) \frac{\partial}{\partial x} \phi(x) \,\,.$$

- a. Consider the wave packet (56) with infinite width, $L = \infty$. Check that the expression (55), for any time t (no need to take the limit), and its hermitian conjugate, give rise to the canonical annihilation and creation operators of the Klein-Gordon model.
- b. There are many very simple conserved charges in involution that can be constructed:

$$P_{(n)} = (-1)^{\frac{n+1}{2}} \int dx \, \pi(x) \left(\frac{\partial}{\partial x}\right)^n \phi(x)$$

for any n positive and odd. Show that these act as

$$[P_{(n)},\phi(x)] = \left(i\frac{\partial}{\partial x}\right)^n \phi(x) , \quad [P_{(n)},\pi(x)] = \left(i\frac{\partial}{\partial x}\right)^n \pi(x) .$$

and that they are indeed conserved and in involution.

c. Other charges are

$$H_{(n)} = (-1)^{\frac{n-1}{2}} \int dx \, \frac{1}{2} \left[\pi(x) \left(\frac{\partial}{\partial t}\right)^n \phi(x) - \phi(x) \left(\frac{\partial}{\partial t}\right)^n \pi(x) \right]$$

for odd positive n, where the time derivatives $\partial/\partial t$ should really be replaced by what is obtained from the equation of motion: $\partial/\partial t = i[H, \cdot]$. Show that these charges have actions

$$[H_{(n)},\phi(x)] = \left(-i\frac{\partial}{\partial t}\right)^n \phi(x) , \quad [H_{(n)},\pi(x)] = \left(-i\frac{\partial}{\partial t}\right)^n \pi(x)$$

with the same meaning for the time derivatives, and that they are indeed conserved and in involution.

d. In similar ways, one can imagine how to construct local conserved charges whose action on the fields $\phi(x)$ and $\pi(x)$ are of the type

$$[Q_{(n,k)},\phi(x)] = \left(i\frac{\partial}{\partial x}\right)^n \left(-i\frac{\partial}{\partial t}\right)^k \phi(x) , \quad [Q_{(n,k)},\pi(x)] = \left(i\frac{\partial}{\partial x}\right)^n \left(-i\frac{\partial}{\partial t}\right)^k \pi(x)$$

with n + k positive and odd. They are not all independent, because of the equations of motion. Indeed, we always have, for instance, $(-\partial^2/\partial t^2 + \partial^2/\partial x^2)\phi(x) = m^2\phi(x)$. In order to get the set of independent charges, it is convenient to use the coordinates z = x - t and $\bar{z} = x + t$. Then, one can combine the charges $Q_{(n,k)}$ to form charges that act through

$$\left(\frac{\partial}{\partial z}\right)^{n'} \left(\frac{\partial}{\partial \bar{z}}\right)^{k'}$$

But clearly, $\partial/\partial z \partial/\partial \bar{z} = m^2/4$ as a consequence of the equations of motion, so that we are left with only $\partial/\partial z$ derivatives or $\partial/\partial \bar{z}$ derivatives. Hence, the independent conserved charges are Q_s for odd (positive or negative) integer s whose action on the fields $\phi(x)$ and $\pi(x)$ are examplified by

$$[Q_s,\phi(x)] = \left(\frac{\partial}{\partial z}\right)^s \phi(x) \ (s>0) \ , \quad [Q_s,\phi(x)] = \left(\frac{\partial}{\partial \bar{z}}\right)^s \phi(x) \ (s<0) \ .$$

Show that under the action of the boost operator B, these charges have spin s, that is, $[B, Q_s] = sQ_s$, and that, using the canonical mode operators that diagonalize the Hamiltonian and momentum operators,

$$H = \int \frac{d\theta}{4\pi} m \cosh(\theta) A^{\dagger}(\theta) A(\theta) , \quad P = \int \frac{d\theta}{4\pi} m \sinh(\theta) A^{\dagger}(\theta) A(\theta) ,$$

the conserved charges are simply written as

$$Q_s = \int \frac{d\theta}{4\pi} e^{s\theta} A^{\dagger}(\theta) A(\theta) \; .$$

In particular, $Q_{\pm 1} = m^{-1}(H \pm P)$.

Exercise 3.2 Explicitly verify that (68) solves (67), and that it has no pole on the physical strip.

– Bibliographic notes —–

Already in 1967, Coleman and Mandula discussed the effects of conserved charges in QFT, showing that in dimensions greater than one any nontrivial (higher-spin) space-time symmetries they would impose the theory to be free:

S. Coleman and J. Mandula, "All Possible Symmetries of the S Matrix", Phys. Rev. 159, 1251 (1967)

C. N. Yang was the first to evaluate an exact nontrivial S-matrix for a field theory model in one dimension

C. N. Yang, "S-matrix for the one-dimensional N-body problem with repulsive or attractive δ -function interaction, Phys. Rev. **168** 1920–1923 (1968)

where a version of the Yang-Baxter equation was introduced. Baxter introduced his version in the context of lattice models, see

R. J. Baxter, "Partition function of the eight-vertex lattice model", Ann. Phys. 70, 193-228 (1972)

However, integrability in relativistic QFT really started with the works of Karowski and Weisz, and the Zamolodchikov brothers:

M. Karowski and P. Weisz, "Exact S matrices and form-factors in (1+1)-dimensional field theoretic models with soliton behavior", Nucl. Phys. **B** 139, 455–476 (1978).

A. B. Zamolodchikov and A. B. Zamolodchikov, "Factorized *S*-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models", Ann. Phys. **120**, 253–291 (1979).

Importantly, Parke showed in 1980 how only two higher-spin conserved charges are sufficient in order to have elasticity (absence of particle production) and factorization of scattering:

S. Parke, "Absence of particle production and factorization of the S matrix in (1+1)-dimensional models", Nucl. Phys. **B** 174, 166–182 (1980)

The ensuing theory of factorized scattering was much developed, especially in Smirnov's book

F. A. Smirnov, "Form factors in completely integrable models of quantum field theory", World Scientific, 1992

A good recent book on the subject is Mussardo's:

G. Mussardo, "Statistical Field Theory: An Introduction to Exactly Solved Models in Statistical Physics", Oxford Graduate Texts, 2009

See also Olalla's bibliographical notes on form factors.

Part B: Approaches ton integrable QFT, and measures of entanglement See Olalla's slides.

36

4 Non-equilibrium quantum steady states

Currents and their fluctuations at criticality (2h)

This section of the lectures is based on parts of the following paper:

D. Bernard and B. Doyon, **Conformal field theory out of equilibrium: a review**, J. Stat. Mech. 2016, 064005 (2016), special issue on Nonequilibrium dynamics in integrable quantum systems, P. Calabrese et al., eds., preprint arXiv:1603.07765.

See Subsection 3.2, Subsection 4.3, and Section 5 (covered as time permits)

5 Hydrodynamic approaches to non-equilibrium steady states Generalized hydrodynamics (3h)