3. Entanglement Entropy in Quantum Field Theory: Some References

3. Let us start with a few fundamental papers regarding measures on entanglement in quantum systems. In particular, the ones below deal with proving that the entanglement entropy is indeed a good measure of entanglement and with the notion of preorder implicit within entanglement and represented through the Rényi entropies:

   C. Bennett, H. Bernstein, S. Popescu and B. Schumacher, Phys. Rev. A53 (1996) 2046-2052.
   D. Jonathan and M. Plenio, Phys. Rev. Lett., 83 (1999) 3566-3569.
   M. A. Nielsen, Phys. Rev. Lett. 83 (1999) 436-439.
   S. Turgut, J. Phys. A40 (2007) 12185.

   Another popular measure of entanglement is the logarithmic negativity, see for instance:

   G. Vidal and R. Werner, Phys. Rev. A65 (2002) 032314.
   M. Plenio, Phys. Rev. Lett. 95 (2005) 090503.
   J. Eisert, PhD-Thesis, arXiv:quant-ph/0610253.

   There are many motivations to study entanglement and also many applications thereof, in particular in the context of the study of black hole entropy. Some of the most fundamental works in the area of entanglement originate from this particular application:

   L. Bombelli, R. K. Koul, J. Lee, and R. D. Sorkin, Phys. Rev. D34 (1986) 373.
   M. Srednicki, Phys. Rev. Lett. 71 (1993) 666-669.
   C. Callan and F. Wilczek, Phys. Lett. B333 (1994) 55-61.

   A review of numerical methods employed to simulate quantum states which employ information about the entanglement properties of those states can be found here:

   R. Orus, Ann. Phys. 349 (2014) 117-158.

   In the context of conformal field theory one of the most important papers is:

   C. Holzhey, F. Larsen and F. Wilczek, Nucl. Phys. B424 (1994) 443-467.

   In this paper the famous formula $S(\ell) \sim \frac{c}{3}\log \ell$ for the entanglement of an interval of length ℓ in an infinite system at criticality was first obtained for unitary CFT. The paper uses the " replica trick" to evaluate partition functions on multi-sheeted Riemann surfaces.

However, the scaling of entanglement in critical quantum spin chains had been studied numerically before Holzhey et al. work and with a focus on its applications to quantum information:

G. Vidal, J. I. Latorre, E. Rico and A. Kitaev, Phys. Rev. Lett. 90 (2003) 227902.

Of particular relevance is also the work of Korepin et al. which showed how the entanglement entropy of one interval in free Fermion systems can be evaluated exactly through exact diagonalization of the reduced density matrix. Their work famously used the properties of Toeplitz determinants:

B.-Q. Jin and V. Korepin, J. Stat. Phys. 116 (2004) 79.

In the work of Holzhey, Larsen and Wilczek the entanglement entropy is a function of the partition function of the system in a "replica" theory, that is n copies of the original model. This partition function takes values on a n-sheeted Riemann surface and the interval of length ℓ becomes a brach cut in this surface. The end points of this branch cut can also be seen as insertion points of local conformal fields. The properties of such fields, which are twist fields associated with the cyclic permutation symmetry of the replica theory, have been studied much before they found an application to the study of measures of entanglement. In fact, they were studied in the context of orbifold CFT. In these papers the twist field is formally defined in CFT and its conformal dimension is evaluated:

V. Knizhnik, Commun. Math. Phys. 112 (1987) 567-590.
L. Dixon, D. Friedan, E. Martinec, and S. Shenker, Nucl. Phys. B282 (1987) 13-73.

In the context of CFT the paper that is most often cited as showing the logarithmic scaling of the entanglement entropy in CFT is:

P. Calabrese and J. Cardy, J. Stat. Mech. 2004 (2004) P06002.

This paper was of course published 10 years after the work of Holzhey et al. which already had found this result. However it had huge influence in the field as it made a whole (different) community aware of how CFT techniques could be successfully employed to evaluate the entanglement entropy of critical and off-critical theories in a variety of situations: at finite temperature, in finite volume and for sub-systems consisting of multiple disconnected intervals. This article also introduced the idea that the replica partition function could be represented as a correlator of certain new fields.

However, if you are reading this paper as a novice in the field, it is worth mentioning that the paper contains a few typos and imprecisions. First, the fields identified in this paper are not well-defined (they are non-local) and are not identified as twist fields in the paper. No connection is made to orbifold theory and as a consequence the computation of the dimension of the fields has some imprecisions (the dimensions found are missing a factor n). The proper QFT definition of these fields and identification of their dimension was presented later on in:

J. L. Cardy, O. A. Castro-Alvaredo and B. Doyon, J. Stat. Phys. (2008) 130 129-168.

Where the focuss was on massive IQFT. Second, Calabrese & Cardy's paper contains one important new result which is a general formula for the entanglement of a subsystem consisting of several disconnected intervals. This formula was however found to be wrong in general (except for free Fermions). This was pointed out by various authors and then investigated further by Calabrese, Cardy and Tonni in a series of papers, particularly for the compactified free Boson (see later).

The notion of branch point twist fields can also be generalised to discrete systems such as quantum spin chains. We did this in:

O. A. Castro Alvaredo and B. Doyon, J. Stat. Mech. (2011) P02001.

In order to evaluate the entanglement entropy of multiple disconnected regions other techniques are needed (for a single interval it is still possible to use conformal maps to map the n-sheeted Riemann surface back to the sphere, this is not possible for more than one interval and of course also not away from criticality). In terms of twist fields what is involved are higher-point correlation functions which are hard to compute, even in CFT. The first attempt to do precisely this was presented in Appendix A of:

M. Headrick, Phys. Rev. D 82 (2010) 126010.

Employing either the computation of higher-point functions of twist fields or of their corresponding partition function on higher genus Riemann surfaces it is possible to make some progress for free theories but even for those, the problem of analytically continuing results from n positive, integer and larger than 2 to $n = 1$ remains unsolved. The following articles study the entanglement entropy of multiple disconnected regions in various theories:

P. Calabrese, J.L. Cardy and E. Tonni, J. Stat. Mech. (2009) P11001; J. Stat. Mech. (2011) P01021.
M.A. Rajabpour and F. Gliozzi, J. Stat. Mech. (2012) P02016.

The main results for the entanglement entropy of one interval have been generalised to theories where the conformal symmetry is spontaneously broken, such non-unitary and non-compact CFT. From the twist field point of view, a new "composite" twist field is introduced and the result of Holzhey et al. is recovered with c replaced by the effective central charge $c_{\mathbf{eff}}$:

D. Bianchini, O. A. Castro-Alvaredo, B. Doyon, E. Levi and F. Ravanini, J. Phys. A48 (2015) 04FT01.

Conformal field theory techniques have also been used recently to study the entanglement entropy of excited states in critical systems:

F. Castilho Alcaraz, M. Ibañez Berganza and G. Sierra, Phys. Rev. Lett. 106 (2011) 201601.
M. Ibañez Berganza, F. Castilho Alcaraz and G. Sierra, J. Stat. Mech. 1201 (2012) P01016.
B. Herwerth, G. Sierra, H.-H. Tu and A.E.B. Nielsen, Phys. Rev. B91 (2015) 235121.

The entanglement between two semi-infinite halves of an infinite off-critical quantum system takes a constant value which is related to the correlation length. For free Fermions the transfer matrix may be computed exactly, an approach which was first employed by Peschel in the study of free theories:

I. Peschel, J. Phys. A36 (2003) L205; J. Stat. Mech. (2004) P06004.

For more complicated (interacting) models the entanglement entropy may be evaluated by employing Baxter's corner transfer matrix approach. This has been done in:

E. Ercolessi, S. Evangelisti and F. Ravanini, Phys. Lett. A374 (2010) 2101-2105.
E. Ercolessi, S. Evangelisti, F. Franchini and F. Ravanini, Phys. Rev. B83 (2011) 012402; Phys. Rev. B85 (2012) 115428.

The corner transfer matrix approach has also been used for non-unitary lattice models, showing once more, the dependence on the effective central charge.

D. Bianchini and F. Ravanini, J. Phys. A49 (2016) 154005.

Most of the work reviewed so far deals with critical continuous systems described by CFT. It is however interesting to go beyond criticality and look at massive QFTs. This is however much more complicated as conformal invariance cannot be used. The only result that was known for some time and first presented in the original work of Calabrese & Cardy (2004) is that the entanglement entropy of one interval whose length is much larger than the correlation length in an infinite system saturates to a constant value, related to the value of the correlation length (this is twice the value of the entanglement between two semi-infinite regions of an infinite system). In order to go beyond this simple case an look at intervals of length which is comparable to the correlation length it was necessary to exploit other methods of QFT. Our approach in:

J.L. Cardy, O.A. Castro-Alvaredo and B. Doyon, J. Stat. Phys. (2008) 130: 129?168,

is based on the use of correlators of branch point twist fields. The name branch point twist field was actually first used in this paper. Once the entanglement entropy is related to correlation functions of twist fields, the form factor programme can be generalised for these field and the entanglement entropy can be evaluated in terms of a form factor expansion. Many results then require the QFT to be integrable, but some of our results have been shown to hold beyond integrability too. In particular, we found that there are universal exponentially decaying corrections to saturation of the entanglement entropy which are characterized by the mass spectrum of the QFT and which take the same form, irrespective of integrability. We generalised our results to non-diagonal scattering:

O. A. Castro Alvaredo and B. Doyon, J. Phys. A41 (2008) 275203,

and beyond integrable QFT:

B. Doyon, Phys. Rev. Lett. 102 (2009) 031602.

The form of the exponential corrections to saturation has been confirmed by numerical work on quantum spin chains:

E. Levi, O.A. Castro-Alvaredo and B. Doyon, Phys. Rev. B88 (2013).
J. Sirker, M. Maiti, N.P. Konstantinidis and N. Sedlmayr, J. Stat. Mech. P10032 (2014).

We have also studied the entanglement of a massive perturbation of a non-unitary QFT, namely the Lee-Yang model:

D. Bianchini, O. A. Castro-Alvaredo and B. Doyon, Nucl. Phys. B896 (2015) 835?880.

The form factor programme for twist fields has been studied for various other diagonal theories. In particular, a complicated diagonal theory with bound states and non-vanishing one particle form factors has been recently studied in:

O. A. Castro-Alvaredo, arXiv:1610.07040.

Recently, much work has been carried out to compute the logarithmic negativity of CFT. This is a different measure of entanglement which also exhibits universal features (such as logarithmic scaling characterized by the central charge). The main effort in this context comes from Calabrese, Cardy and Tonni:

P. Calabrese, J. Cardy and E. Tonni, Phys. Rev. Lett. 109 (2012) 130502.
P. Calabrese, J. Cardy and E. Tonni, J. Stat. Mech. 2013 (2013) P02008.

In the context of the AdS/CFT correspondence there is also an elegant expression for the entanglement entropy which relates to the area of a minimal surface. The most famous contribution in this context is by Ryu & Takayanagi:

S. Ryu and T. Takayanagi, Phys. Rev. Lett. 96, 181602.

Finally, it turns out that logarithmic scaling of the entanglement entropy and the logarithmic negativity are features not only of CFT but also of other types of "criticality". In particular, it is found in theories where the ground state is highly degenerate and there is spontaneous symmetry breaking. In this case, the coefficient of the logarithmic divergence is not a central charge but the number of Goldstone bosons which can also be interpreted as a geometric dimension which may be non-integer. A model where such behaviour has been observed is the ferromagnetic XXX chain and its higher spin generalisations:

V. Popkov and M. Salerno, Phys. Rev. A71 (20005) 012301.
V. Popkov, M. Salerno and G. Schütz, Phys. Rev. A72 (2005) 032327.

We proved and generalised these results to a large class of states employing our spin chain twist fields:

O. A. Castro Alvaredo and B. Doyon, Phys. Rev. Lett. 108 (2012) 120401.
O. A. Castro-Alvaredo and B. Doyon, J. Stat. Mech. (2013) P02016.

Interestingly, the same kind of behaviour is also observed for "random" spin chains, that is spin chain models with random couplings.

G. Refael and J. E. Moore, Phys. Rev. Lett. 93 (2004) 260602.
M. Fagotti, P. Calabrese and J.E. Moore, Phys. Rev. B83 (2011) 045110.
P. Ruggiero, V. Alba and P. Calabrese, Phys. Rev. B94 (2016) 035152.

G. Ramírez, J. Rodríguez-Laguna and G. Sierra, J. Stat. Mech. 1506 (2015) P06002.