1. Thermodynamic Bethe Ansatz: Some References

1. The TBA may be seen as a thermodynamic (infinite volume) version of the Bethe equations. In this limit the Bethe roots entering the Bethe equations become continuously distributed, according to some density, as we have seen. In Zamolodchikov's formulation the TBA is also relativistic in the sense that the relativistic scattering phase is involved, but this is not necessary. One can also write TBA equations for non-relativistic systems. There are many references regarding the Bethe equations particularly in a discrete system known as the Heisenberg chain:

   W. Heisenberg, Zeitschrift für Physik 49 (1928) 619.
   H. Bethe, Zeitschrift für Physik 71 (1931) 205.
   R. Orbach, Phys. Rev. 112 (1958) 309.
   C.N. Yang and C.P. Yang, Phys. Rev. 150 (1966) 327.

   Yang & Yang described the exact solution of the Bethe equations by a method known as coordinate Bethe ansatz. This was later on generalized to what is today known as the algebraic Bethe ansatz:

   L.D. Faddeev, E.K. Sklyanin and L.A. Takhtajan, Theor. Math. Phys. 40 (1979) 688,

   and has been used very successfully to for instance compute correlation functions of quantum spin chains. See for instance:

   N. Kitanine, J. M. Maillet and V. Terras, Nucl. Phys. B567 (2000) 554-582.

   The thermodynamic Bethe ansatz as presented in the lecture was first proposed and used for the Lee-Yang model and the scaling 3-state Potts model in:

   Al. B. Zamolodchikov, Nucl. Phys. B342 (1990) 695-720.

   Shortly after there were other papers, especially by Klassen & Melzer where the same approach was used for many different models. Their papers provide also a very detailed explanation of the method:

   T. Klassen and E. Melzer, Nucl. Phys. B338 (1990) 485-528.
   T. Klassen and E. Melzer, Nucl. Phys. B350 (1991) 635-689.

   Note that Klassen & Melzer's work seems early than Zamolodchikov's but the latter was known in Pre-print form by 1989.

In these papers the issue of how to fix the particles' statistics is discussed and it is determined that if $S_{aa}(0) = 1$ the statistics is Bosonic and if $S_{aa}(0) = -1$ then it is Fermionic.

The TBA as presented can be generalised in many ways. For instance, to consider more general types of statistics such as Haldane statistics:

F.D.M. Haldane, Phys. Rev. Lett. 67 (1991) 937.
A. Bytsko and A. Fring, Nucl. Phys. B532 (1998) 588-608.

To consider theories where parity invariance is broken, that is, in general $S_{ab}(\theta) \neq S_{ba}(\theta)$. A prime example of such theories is a family of models knowns as Homogenous sine-Gordon models which were studied in great detail in O. A. Castro-Alvaredo's PhD thesis and subsequent work. Their S-matrices were computed in:

J.L. Miramontes and C.R. Fernández-Pousa, Phys. Lett. B472 (2000) 392-401.

and a TBA analysis was carried out in:

O.A. Castro-Alvaredo, A. Fring, C. Korff. J.L. Miramontes, Nucl. Phys. B575 (2000) 535-560.

Finally, it is of course interesting to formulate the TBA equations for integrable models with non-diagonal scattering , such as the famous sine-Gordon model. This is considerably more complicated but has been done in:

C. Destri, H.J. de Vega, Nucl. Phys. B438 (1995)413-454; Nucl. Phys. B504 (1997) 621-664.

The fact that the ground state energy of a conformal field theory is proportional to the central charge at finite volume (also known a Casimir effect) was shown in:

I. Affleck, Phys. Rev. Lett. 56 (1986) 746-748.
H. Blöte, J. Cardy, M. Nightingale, Phys. Rev. Lett. 56 (1986) 742-745.

Almost simultaneously it was shown that the central charge should in general be replaced by the effective central charge which may be different for instance in non-unitary models.

C. Itzykson, H. Saleur, J. Zuber, Europhys. Lett. 2 (1986) 91.

The scaling function $c(R)$ has been computed and studied for many models. In most cases it just monotonically flows from zero in the IR to the value of $c_{\mathbf{eff}}$ in the UV.

There are however some models where more interesting behaviour is observed, particularly when the flow from the IR to the UV passes near several critical points. In such cases the c-functions develops staircase patterns, first observed in the roaming trajectories model of Zamolodchikov:

Al. B. Zamolodchikov, J. Phys. A39 (2006) 12847?61. [Note that this article appeared as a pre-print in 1991]

and its generalisations:

P. Dorey and F. Ravanini, Int. J. Mod. Phys. A8 (1993) 873-894; Nucl. Phys. B406 (1993) 708-726.

As we have seen, the homogenous sine-Gordon models also give rise to staircase patterns which are related to the presence of unstable particles in the spectrum. Some examples have been studied here:

O.A. Castro-Alvaredo, A. Fring, C. Korff. J. Miramontes, Nucl. Phys. B575 (2000) 535-560.
P. Dorey and J. L. Miramontes, Nucl. Phys. B697 (2004) 405-461.
O. A. Castro-Alvaredo, A. Fring and J. Dreißig, Eur. Phys. J. C35 (2004) 393-411.

We have also mentioned that the solutions to the constant TBA equations have very interesting algebraic structures and for many theories are related to the Weyl characters of particular Lie algebras. The constant TBA equations have been studied in much detail in:

A. Kuniba, Nucl. Phys. B389 (1993) 209.
A. Kuniba and T. Nakanishi, Mod. Phys. 7 (1992) 3487.
A. Kuniba, T. Nakanishi and J. Suzuki, Mod. Phys. 8 (1993) 1649; Int. J. Mod. Phys. 9 (1994) 5215; Int. J. Mod. Phys. 9 (1994) 5267.

Finally, the TBA equations can be recast in the form of Y-systems. This allows us to move from solving integro-differential equations to solving difference equations for functions with particular asymptotic properties. Y-systems and their solutions have been studied for many models and continue to be popular today. Some early examples can be found in:

Al. B. Zamolodchikov, Phys. Lett. B261 (1991) 57.
Al. B. Zamolodchikov, Nucl. Phys. B358 (1991) 524.

In the first paper the A_n minimal Toda theories were studied. It was found that the Y-functions had a periodicity which is related to the dimension of the perturbing field.