Form Factor Programme: Solutions to Exercises

1. **Minimal form factor**
This is a very simple exercise. To prove the first identity $f_{ab}(\theta) = S_{ab}(\theta) f_{ab}(-\theta)$ holds if
$$\frac{g_{ab}(t)}{\sinh t} \sin^2\left(\frac{it}{2}\left(1+\frac{i\theta}{\pi}\right)\right) = g_{ab}(t) \sinh \frac{t\theta}{\pi} + \frac{g_{ab}(t)}{\sinh t} \sin^2\left(\frac{it}{2}\left(1-\frac{i\theta}{\pi}\right)\right)$$
$$= \frac{g_{ab}(t)}{\sinh t}\left(\sinh t \sinh\frac{t\theta}{\pi} + \sin^2\left(\frac{it}{2}\left(1-\frac{i\theta}{\pi}\right)\right)\right)$$
Since $\sin^2(i(a+b)) = \sin^2(i(a-b)) - \sinh(2a)\sinh(2b)$ we have that
$$\sin^2\left(\frac{it}{2}\left(1-\frac{i\theta}{\pi}\right)\right) = \sin^2\left(\frac{it}{2}\left(1+\frac{i\theta}{\pi}\right)\right) - \sinh t \sinh\frac{t\theta}{\pi}$$
which proves the identity. Similarly, to prove that $f_{ab}(\theta) = f_{ab}(-\theta + 2\pi i)$ we need to show that
$$\sin^2\left(\frac{it}{2}\left(1+\frac{i\theta}{\pi}\right)\right) = \sin^2\left(\frac{it}{2}\left(1+\frac{i(-\theta+2\pi i)}{\pi}\right)\right) = \sin^2\left(\frac{it}{2}\left(-1-\frac{i\theta}{\pi}\right)\right),$$
which clearly also holds!

2. ***c*-function in the Ising model**
We have that
$$F_0^\Theta := \langle 0|\Theta|0\rangle = 2\pi m^2 \qquad F_2^\Theta(\theta_1,\theta_2) := -2\pi i m^2 \sinh\frac{\theta_1-\theta_2}{2}.$$
and the c-theorem tell us that there is a function $c(r)$ defined as
$$c(r) = \frac{3}{2}\int_r^\infty ds\, s^3 \langle 0|\Theta(0)\Theta(s)|0\rangle_c.$$
The Ising model is a very simple theory (essentially a free Fermion) and the form factors of all its local fields are known exactly (see question 4 also). Because the stress energy tensor only has two non-vanishing form factors, this is a rare example in which the form factor expansion of the two-point function is actually exact after including only two-particle form factors (higher particle terms are all vanishing). Using the expansion we saw in the lecture and considering the connected correlator
$$\langle 0|\Theta(0)\Theta(r)|0\rangle_c = \langle 0|\Theta(0)\Theta(r)|0\rangle - \langle 0|\Theta|0\rangle^2$$
we have that
$$\langle 0|\Theta(0)\Theta(r)|0\rangle_c = \frac{1}{2(2\pi)^2}\int_{-\infty}^\infty d\theta_1 \int_{-\infty}^\infty d\theta_2 |F_2^\Theta(\theta_1,\theta_2)|^2 e^{-rm(\cosh\theta_1+\cosh\theta_2)}$$
$$= \frac{m^4}{2}\int_{-\infty}^\infty d\theta_1\int_{-\infty}^\infty d\theta_2 \sinh^2\left(\frac{\theta_1-\theta_2}{2}\right) e^{-2rm\cosh\frac{\theta_1-\theta_2}{2}\cosh\frac{\theta_1+\theta_2}{2}}.$$

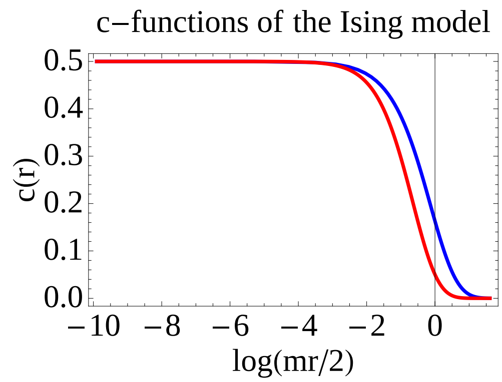

Figure 1: The blue curve is the TBA scaling function. The red curve is Zamolodchikov's c-function.

Changing variables to $x = \theta_1 - \theta_2$ and $y = \theta_1 + \theta_2$ we have

$$\langle 0|\Theta(0)\Theta(r)|0\rangle_c = \frac{m^4}{4}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \sinh^2\frac{x}{2} e^{-2rm\cosh\frac{x}{2}\cosh\frac{y}{2}}.$$

and integrating in the variable y we have

$$\langle 0|\Theta(0)\Theta(r)|0\rangle_c = m^4 \int_{-\infty}^{\infty} dx \sinh^2\frac{x}{2} K_0(2mr\cosh\frac{x}{2}) = 2m^4 \int_0^{\infty} dx \sinh^2\frac{x}{2} K_0(2mr\cosh\frac{x}{2})$$

$$= 4m^4 \int_1^{\infty} du \sqrt{u^2-1} K_0(2mru)$$

where we introduced the new variable $u = \cosh\frac{x}{2}$. Thus, the c-function is

$$c(r) = 6m^4 \int_r^{\infty} ds\, s^3 \int_1^{\infty} du \sqrt{u^2-1} K_0(2msu).$$

The integral is s can be performed exactly to:

$$\int_r^{\infty} ds\, s^3 K_0(2msu) = \frac{r^2}{(2mu)^2}\left(2murK_1(2mur) + 2K_2(2mur)\right).$$

Thus we obtain

$$c(r) = \frac{3(mr)^2}{2}\int_1^{\infty} \frac{du}{u^2}\sqrt{u^2-1}\left(2murK_1(2mur) + 2K_2(2mur)\right).$$

Note that $c(r)$ is a function of mr (as in the TBA). That is why we call it a "scaling" function. It is invariant under simultaneous scaling of the mass and the distance!

From this expression it is easy to extract the value $c(0)$. We can expand the Bessel functions about small mr:

$$2murK_1(2mur) + 2K_2(2mur) \sim \frac{1}{(mur)^2} + \textbf{constant},$$

giving

$$c(0) = \frac{3}{2}\int_1^\infty \frac{du}{u^4}\sqrt{u^2-1} = \frac{1}{2}.$$

A plot of this function and the TBA scaling function for the Ising model in logarithmic scale is presented in Fig. 1.

3. **Cumulant expansion of two-point functions**

The first part of the question involves comparing the form factor expansion of two point functions with a similar expansion of the logarithm of the two-point function. We have already see that if $\mathcal{O}$ is self-conjugate then we can expand the two-point function as

$$\frac{\langle 0|\mathcal{O}(0)\mathcal{O}(r)|0\rangle}{\langle 0|\mathcal{O}|0\rangle^2} = \langle 0|\mathcal{O}|0\rangle^{-2}\sum_{k=0}^\infty \int_{-\infty}^\infty \frac{d\theta_1\ldots d\theta_k}{k!(2\pi)^k}|F_k^{\mathcal{O}}(\theta_1,\ldots,\theta_k)|^2 e^{-mr\sum_{j=1}^k \cosh\theta_j}$$

The question states that this same ratio may be written as

$$\frac{\langle 0|\mathcal{O}(0)\mathcal{O}(r)|0\rangle}{\langle 0|\mathcal{O}|0\rangle^2} = \exp\left[\sum_{k=1}^\infty \int_{-\infty}^\infty \frac{d\theta_1\ldots d\theta_k}{k!(2\pi)^k} h_k(\theta_1,\ldots,\theta_k)e^{-mr\sum_{j=1}^k \cosh\theta_j}\right].$$

So we just need to compare the two formulae term-by-term by collecting together terms that have the same number of integrals (or the same mr-dependance). The exponential above can be expanded in the usual way

$$\begin{aligned}
&\exp\left[\sum_{k=1}^\infty \int_{-\infty}^\infty \frac{d\theta_1\ldots d\theta_k}{k!(2\pi)^k} h_k(\theta_1,\ldots,\theta_k)e^{-mr\sum_{j=1}^k \cosh\theta_j}\right] \\
&= 1 + \sum_{k=1}^\infty \int_{-\infty}^\infty \frac{d\theta_1\ldots d\theta_k}{k!(2\pi)^k} h_k(\theta_1,\ldots,\theta_k)e^{-mr\sum_{j=1}^k \cosh\theta_j} \\
&+\frac{1}{2}\left[\sum_{k=1}^\infty \int_{-\infty}^\infty \frac{d\theta_1\ldots d\theta_k}{k!(2\pi)^k} h_k(\theta_1,\ldots,\theta_k)e^{-mr\sum_{j=1}^k \cosh\theta_j}\right]^2 + \cdots \\
&= 1 + \int_{-\infty}^\infty \frac{d\theta_1}{2\pi}h_1(\theta_1)e^{-rm\cosh\theta_1} + \frac{1}{2}\left(\int_{-\infty}^\infty \frac{d\theta_1}{2\pi}h_1(\theta_1)e^{-rm\cosh\theta_1}\right)^2 \\
&+\frac{1}{3!}\left(\int_{-\infty}^\infty \frac{d\theta_1}{2\pi}h_1(\theta_1)e^{-rm\cosh\theta_1}\right)^3 + \cdots + \int_{-\infty}^\infty\int_{-\infty}^\infty \frac{d\theta_1 d\theta_2}{2(2\pi)^2}h_2(\theta_1,\theta_2)e^{-rm(\cosh\theta_1+\cosh\theta_2)} + \cdots
\end{aligned}$$

Comparing with the form factor expansion we have that

$$h_1(\theta) = \frac{|F_1^{\mathcal{O}}(\theta)|^2}{\langle 0|\mathcal{O}|0\rangle^2},$$

$$h_2(\theta_1,\theta_2) + h_1(\theta_1)h_1(\theta_2) = \frac{|F_2^{\mathcal{O}}(\theta_1,\theta_2)|^2}{\langle 0|\mathcal{O}|0\rangle^2},$$

$$h_3(\theta_1,\theta_2,\theta_3)+h_1(\theta_1)h_1(\theta_2)h_1(\theta_3)+h_1(\theta_1)h_2(\theta_2,\theta_3)+h_1(\theta_2)h_2(\theta_1,\theta_3)+h_1(\theta_3)h_2(\theta_1,\theta_2) = \frac{|F_3^{\mathcal{O}}(\theta_1,\theta_2,\theta_3)|^2}{\langle 0|\mathcal{O}|0\rangle^2},$$

and so on.

Consider again the cumulant expansion:

$$\log\left(\frac{\langle 0|\mathcal{O}(0)\mathcal{O}(r)|0\rangle}{\langle 0|\mathcal{O}|0\rangle^2}\right) = \sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\frac{d\theta_1\dots d\theta_k}{k!(2\pi)^k}h_k(\theta_1,\dots,\theta_k)e^{-mr\sum_{j=1}^k\cosh\theta_j}.$$

If the field $\mathcal{O}$ is spinless we know that all its form factors are functions of rapidity differences. The same is true for the functions h_k. Let us introduce new variables:

$$\beta_1 = \theta_1, \quad \beta_i = \theta_i - \theta_1 \quad \textbf{for} \quad i = 2,\dots,k.$$

Then

$$h_k(\theta_1,\theta_2,\dots,\theta_k) = h_k(\beta_1,\beta_2+\beta_1,\dots,\beta_k+\beta_1) = h_k(0,\beta_2,\dots,\beta_k)$$

The cumulant expansion can then be rewritten as

$$\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\frac{d\beta_1\dots d\beta_k}{k!(2\pi)^k}h_k(0,\beta_2,\dots,\beta_k)e^{-mr\cosh\beta_1-mr\sum_{j=2}^k\cosh(\beta_j+\beta_1)}.$$

The variable β_1 now only features in the exponential and so the integral can be carried out. In fact

$$\cosh\beta_1+\sum_{j=2}^k\cosh(\beta_j+\beta_1) = \cosh\beta_1(1+\sum_{j=2}^k\cosh\beta_j)+\sinh\beta_1\sum_{j=2}^k\sinh\beta_j.$$

So we need to carry out an integral of the form

$$\int_{-\infty}^{\infty}e^{-a\cosh\beta-b\sinh\beta}d\beta = 2K_0(\sqrt{a^2-b^2}) \quad \textbf{for} \quad a>b.$$

The cumulant expansion can then be written as

$$\log\left(\frac{\langle 0|\mathcal{O}(0)\mathcal{O}(r)|0\rangle}{\langle 0|\mathcal{O}|0\rangle^2}\right) = 2\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\frac{d\beta_2\dots d\beta_k}{k!(2\pi)^k}h_k(0,\beta_2,\dots,\beta_k)K_0(mrd(\beta_2,\dots,\beta_k)).$$

with

$$d(\beta_2,\dots,\beta_k) = \sqrt{(1+\sum_{j=2}^k\cosh\beta_j)^2-(\sum_{j=2}^k\sinh\beta_j)^2}.$$

For small mr we may expand the Bessel function as

$$K_0(x)\sim -\log x+\log 2-\gamma_E,$$

where $\gamma_E = 0.57721566$ is the Euler-Mascheroni constant. The cumulant expansion is then approximated by

$$\log\left(\frac{\langle 0|\mathcal{O}(0)\mathcal{O}(r)|0\rangle}{\langle 0|\mathcal{O}|0\rangle^2}\right) \approx -2\log(mr)\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\frac{d\beta_2\ldots d\beta_k}{k!(2\pi)^k}h_k(0,\beta_2,\ldots,\beta_k)$$

$$-2\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\frac{d\beta_2\ldots d\beta_k}{k!(2\pi)^k}h_k(0,\beta_2,\ldots,\beta_k)\left(\log(d(\beta_2,\ldots,\beta_k)) - \log 2 + \gamma_E\right) + \cdots$$

Comparing with the expected short-distance behaviour

$$\log\left(\frac{\langle 0|\mathcal{O}(0)\mathcal{O}(r)|0\rangle}{\langle 0|\mathcal{O}|0\rangle^2}\right) \approx -4\Delta_{\mathcal{O}}\log r - 2\log\langle\mathcal{O}\rangle + \cdots$$

we see that

$$\Delta_{\mathcal{O}} = \frac{1}{2}\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\frac{d\beta_2\ldots d\beta_k}{k!(2\pi)^k}h_k(0,\beta_2,\ldots,\beta_k),$$

as expected. In addition we can even identify the constant correction as

$$\log\langle\mathcal{O}\rangle = \sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\frac{d\beta_2\ldots d\beta_k}{k!(2\pi)^k}h_k(0,\beta_2,\ldots,\beta_k)\left(\log(d(\beta_2,\ldots,\beta_k)) - \log 2 + \gamma_E\right)$$

which means it is in principle possible to obtain the vacuum expectation value of an operator from its form factors.

The first paper I know of where the cumulant expansion is used and the formula for the dimension given is: F. Smirnov, Nucl. Phys. B 337 (1990) 156-180. The formula for the vacuum expectation value appeared for the first time (as far as I know) in H. Babujian, M. Karowski, Int. J. Mod. Phys. A 19 (2004) 34-49. Both the formulae for $\Delta_{\mathcal{O}}$ and for the vacuum expectation value have been used by many people, including in some of our work.

4. This seems like a long problem but it is in fact very simple. Let us consider just one case:

$$\lim_{\lambda\to\infty} F_{2k}^{\mu}(\theta_1+\lambda,\ldots\theta_p+\lambda,\theta_{p+1},\ldots\theta_{2k}) = \lim_{\lambda\to\infty} i^k\langle 0|\mu|0\rangle\left[\prod_{1\leq i<j\leq p}\tanh\frac{\theta_i-\theta_j}{2}\right]$$

$$\times\left[\prod_{p+1\leq i<j\leq 2k}\tanh\frac{\theta_i-\theta_j}{2}\right]\left[\prod_{i=1}^{p}\prod_{j=p+1}^{2k}\tanh\frac{\theta_i+\lambda-\theta_j}{2}\right]$$

$$= i^k\langle 0|\mu|0\rangle\left[\prod_{1\leq i<j\leq p}\tanh\frac{\theta_i-\theta_j}{2}\right]\left[\prod_{p+1\leq i<j\leq 2k}\tanh\frac{\theta_i-\theta_j}{2}\right]$$

All the tanh terms containing the λ shift tend to 1 so what remains is the product of two form factors of either μ if p is even or of σ if p is odd. The same reasoning shows all other cases. The Ising model provides the simplest theory which possesses an internal $\mathbb{Z}_2$ symmetry. This is behind the fact that clustering can mix the form factors of two different fields. An argument to show the origin of the property of cluster decomposition was given in G. Delfino, P. Simonetti and J. L. Cardy, Phys. Lett. B 387 (1996) 327 under the assumption of no internal symmetries.

5. **Δ-sum rule in the Ising model**
The Delta-sum rule states that:

$$\Delta_\mu = -\frac{1}{2\langle 0|\mu|0\rangle}\int_0^\infty ds\, s\, \langle\Theta(0)\mu(s)\rangle_c$$

Once more because the stress-energy tensor only has two non-vanishing form factors, the form factor expansion of the correlation function only contains one term and the value of Δ_μ should be obtained exactly. The calculation is quite similar as for Zamolodchikov"s c-function. We have that

$$\langle 0|\Theta(0)\mu(r)|0\rangle_c = \langle 0|\Theta(0)\mu(r)|0\rangle - \langle 0|\Theta|0\rangle\langle 0|\mu|0\rangle$$

we have that

$$\langle 0|\Theta(0)\mu(r)|0\rangle_c = \frac{1}{2(2\pi)^2}\int_{-\infty}^{\infty} d\theta_1 \int_{-\infty}^{\infty} d\theta_2 F_2^\Theta(\theta_1,\theta_2)F_2^\mu(\theta_1,\theta_2)^* e^{-rm(\cosh\theta_1+\cosh\theta_2)}$$

$$= -\frac{m^2\langle 0|\mu|0\rangle}{4\pi}\int_{-\infty}^{\infty} d\theta_1 \int_{-\infty}^{\infty} d\theta_2 \sinh\left(\frac{\theta_1-\theta_2}{2}\right)\tanh\left(\frac{\theta_1-\theta_2}{2}\right)e^{-2rm\cosh\frac{\theta_1-\theta_2}{2}\cosh\frac{\theta_1+\theta_2}{2}}.$$

Changing variables to $x=\theta_1-\theta_2$ and $y=\theta_1+\theta_2$ we have

$$\langle 0|\Theta(0)\mu(r)|0\rangle_c = -\frac{m^2\langle 0|\mu|0\rangle}{8\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{\sinh^2\frac{x}{2}}{\cosh\frac{x}{2}} e^{-2rm\cosh\frac{x}{2}\cosh\frac{y}{2}}.$$

and integrating in the variable y we have

$$\langle 0|\Theta(0)\mu(r)|0\rangle_c = -\frac{m^2\langle 0|\mu|0\rangle}{2\pi}\int_{-\infty}^{\infty} dx \frac{\sinh^2\frac{x}{2}}{\cosh\frac{x}{2}} K_0(2mr\cosh\frac{x}{2})$$

$$= -\frac{m^2\langle 0|\mu|0\rangle}{\pi}\int_0^\infty dx\frac{\sinh^2\frac{x}{2}}{\cosh\frac{x}{2}}K_0(2mr\cosh\frac{x}{2}) = -\frac{2m^2\langle 0|\mu|0\rangle}{\pi}\int_1^\infty \frac{du}{u}\sqrt{u^2-1}K_0(2mru).$$

where we introduced the new variable $u=\cosh\frac{x}{2}$. The Δ-sum rule is:

$$\Delta_\mu = \frac{m^2}{\pi}\int_0^\infty ds\, s \int_1^\infty \frac{du}{u}\sqrt{u^2-1}K_0(2msu).$$

The integral in s is simply:

$$\int_0^\infty ds\, s\, K_0(2msu) = \frac{1}{(2mu)^2}.$$

So,

$$\Delta_\mu = \frac{1}{4\pi}\int_1^\infty \frac{du}{u^3}\sqrt{u^2-1} = \frac{1}{16}.$$

This calculation was performed first in G. Delfino, P. Simonetti and J.L. Cardy, Phys. Lett. B 387 (1996) 327 where the Δ-sum rule was originally proposed.

6. **Form factors of the sinh-Gordon model**
   The kinematic residue equation is

$$\lim_{\bar\theta\to\theta}(\bar\theta-\theta)F_{k+2}(\bar\theta+i\pi,\theta,\theta_1,\ldots,\theta_k) = i(1-\prod_{j=1}^k S(\theta-\theta_j))F_k(\theta_1,\ldots,\theta_k)$$

Employing the ansatz we have that

$$F_{k+2}(\bar\theta+i\pi,\theta,\theta_1,\ldots,\theta_k) = H_{k+2}Q_{k+2}(-\bar x,x,x_1,\ldots,x_k)\left[\prod_{1\le i<j\le k}\frac{F_{\mathbf{min}}(\theta_i-\theta_j)}{x_i+x_j}\right]$$

$$\times\left[\prod_{j=1}^k\frac{F_{\mathbf{min}}(\theta-\theta_j)F_{\mathbf{min}}(\bar\theta+i\pi-\theta_j)}{(x+x_j)(-\bar x+x_j)}\right]\frac{F_{\mathbf{min}}(i\pi)}{-\bar x+x}$$

with $\bar x = e^{\bar\theta}$ and $x = e^\theta$. The kinematic residue equation then becomes

$$\lim_{\bar\theta\to\theta}(\bar\theta-\theta)H_{k+2}Q_{k+2}(-\bar x,x,x_1,\ldots,x_k)\frac{F_{\mathbf{min}}(i\pi)}{-\bar x+x}\prod_{j=1}^k\frac{F_{\mathbf{min}}(\theta-\theta_j)F_{\mathbf{min}}(\bar\theta+i\pi-\theta_j)}{(x+x_j)(-\bar x+x_j)}$$

$$= i(1-\prod_{j=1}^k S(\theta-\theta_j))H_kQ_k(\theta_1,\ldots,\theta_k).$$

On the left hand side of the equation, the only pole comes from the term $1/(x-\bar x)$. We can easily compute

$$\lim_{\bar\theta\to\theta}\frac{\bar\theta-\theta}{x-\bar x} = -\frac{1}{x}.$$

We can also rewrite the factor $(1-\prod_{j=1}^k S(\theta-\theta_j))$ in terms of the x-variables by using:

$$S(\theta) = \left(\frac{x-i}{x+i}\right)^2 \quad \textbf{for} \quad x = e^\theta,$$

so

$$S(\theta-\theta_i) = \left(\frac{x-ix_i}{x+ix_i}\right)^2.$$

Also we can use the information given to write

$$F_{\mathbf{min}}(\theta-\theta_j)F_{\mathbf{min}}(\bar\theta+i\pi-\theta_j) = \frac{x^2-x_j^2}{(x+ix_j)^2}.$$

This allows for various simplifications reducing the equation to

$$F_{\mathbf{min}}(i\pi)(-1)^k H_{k+2}Q_{k+2}(-x,x,x_1,\ldots,x_k)$$

$$= -ix\left(\prod_{j=1}^{k}(x+ix_j)^2 - \prod_{j=1}^{k}(x-ix_j)^2\right) H_k Q_k(x_1,\ldots,x_k)$$

The products above can be expressed in terms of elementary symmetric polynomials by using

$$\prod_{j=1}^{k}(x+x_j) = \sum_{j=0}^{k} x^{k-j}\sigma_j^{(k)},$$

This means that

$$\prod_{j=1}^{k}(x+ix_j)^2 - \prod_{j=1}^{k}(x-ix_j)^2 = (-1)^k\left[\sum_{j=0}^{k}\sum_{p=0}^{k}((-ix)^{2k-j-p} - (ix)^{2k-j-p})\sigma_j^{(k)}\sigma_p^{(k)}\right]$$

$$= (-1)^k\left[\sum_{j=0}^{k}\sum_{p=0}^{k} x^{2k-j-p}((-1)^{p+j}2i\sin(\frac{\pi(2k-j-p)}{2}))\sigma_j^{(k)}\sigma_p^{(k)}\right]$$

$$= 2i\sum_{j=0}^{k}\sum_{p=0}^{k} x^{2k-j-p}(-1)^{p+j+1}\sin(\frac{\pi(j+p)}{2})\sigma_j^{(k)}\sigma_p^{(k)}.$$

Since

$$\sin\frac{\pi(j+p)}{2} = \sin\frac{\pi j}{2}\cos\frac{\pi p}{2} + \cos\frac{\pi j}{2}\sin\frac{\pi p}{2} \quad \textbf{for} \quad j,p\in\mathbb{Z}.$$

We find that the sum above can be fully factorised as

$$= 4i\left(\sum_{j=0}^{k}(-1)^{j+1}x^{k-j}\sin\frac{\pi j}{2}\sigma_j^{(k)}\right)\left(\sum_{p=0}^{k}(-1)^p x^{k-p}\cos\frac{\pi p}{2}\sigma_p^{(k)}\right) := 4iD_k(x,x_1,\ldots,x_k).$$

Therefore we finally have

$$F_{\mathbf{min}}(i\pi)(-1)^k H_{k+2}Q_{k+2}(-x,x,x_1,\ldots,x_k) = 4xD_k(x,x_1,\ldots,x_k)H_kQ_k(x_1,\ldots,x_k).$$

We may write these equations as

$$(-1)^k Q_k(-x,x,x_1,\ldots,x_k) = xD_k(x,x_1,\ldots,x_k),$$

and

$$H_{k+2} = \frac{4H_k}{F_{\mathbf{min}}(i\pi)}.$$

This result was first derived in A. Fring, G. Mussardo and P. Simonetti, Nucl. Phys. B393 (1993) 413.