Thermodynamic Bethe Ansatz: Solutions to Exercises

1. **Derivation of the TBA equations**
This derivation is not presented in any of the classic papers on TBA and it is a rather nice calculation. Here I will consider the simplest case of a diagonal theory with a single particle of mass m. You can easily generalise this to more complicated models. Recall the equations:
$$\rho(\theta) = \frac{m}{2\pi}\cosh\theta + \varphi \star \rho^{(r)}(\theta), \tag{1}$$
and the formulae for the total energy and entropy:
$$h(\rho^{(r)}) = m\int_{-\infty}^{\infty} d\theta\, \rho^{(r)}(\theta)\cosh\theta$$
$$s(\rho,\rho^{(r)}) = \int_{-\infty}^{\infty} d\theta\, (\mp\rho\log\rho - \rho^{(r)}\log\rho^{(r)} - (\rho\pm\rho^{(r)})\log(\rho\pm\rho^{(r)})) \quad \textbf{for Bosons/Fermions}$$
Recall also the definition:
$$\frac{\rho^{(r)}(\theta)}{\rho(\theta)} = \frac{1}{e^{\epsilon(\theta)}\mp 1} \quad \textbf{for Bosons/Fermions} \tag{2}$$
Thermodynamic equilibrium requires that the free energy $f(\rho,\rho^{(r)}) = h(\rho^{(r)}) - Ts(\rho,\rho^{(r)})$ is minimized, that is the functional derivative:
$$\frac{\delta f}{\delta\rho^{(r)}} = 0.$$
Since all quantities involved depend on ρ and $\rho^{(r)}$ and these depend on each other, we will need in particular to compute the functional derivative: $\frac{\delta\rho(\theta)}{\delta\rho^{(r)}(\beta)}$. This can be computed by differentiating equation (1) and employing the definition of the convolution:
$$\frac{\delta\rho(\theta)}{\delta\rho^{(r)}(\beta)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\varphi(\theta-\omega)\frac{\delta\rho^{(r)}(\omega)}{\delta\rho^{(r)}(\beta)}$$
Using $\frac{\delta\rho^{(r)}(\omega)}{\delta\rho^{(r)}(\beta)} = \delta(\omega-\beta)$ we find
$$\frac{\delta\rho(\theta)}{\delta\rho^{(r)}(\beta)} = \frac{1}{2\pi}\varphi(\theta-\beta).$$
Taking the derivative of $f(\rho,\rho^{(r)})$ we have
$$\frac{\delta f(\rho,\rho^{(r)})}{\delta\rho^{(r)}(\beta)} = \frac{\delta h(\rho^{(r)})}{\delta\rho^{(r)}(\beta)} - T\frac{\delta s(\rho,\rho^{(r)})}{\delta\rho^{(r)}(\beta)} = 0,$$
with
$$\frac{\delta h(\rho^{(r)})}{\delta\rho^{(r)}(\beta)} = m\int_{-\infty}^{\infty} d\theta\,\frac{\delta\rho^{(r)}(\theta)}{\delta\rho^{(r)}(\beta)}\cosh\theta = m\cosh\beta.$$

and

$$\frac{\delta s(\rho,\rho^{(r)})}{\delta\rho^{(r)}(\beta)} = \int_{-\infty}^{\infty} d\theta\,\frac{\delta}{\delta\rho^{(r)}(\beta)}(\mp\rho\log\rho - \rho^{(r)}\log\rho^{(r)} \pm (\rho\pm\rho^{(r)})\log(\rho\pm\rho^{(r)}))$$

$$= \int_{-\infty}^{\infty} d\theta\,\Big(\mp\frac{\delta\rho(\theta)}{\delta\rho^{(r)}(\beta)}(\log\rho(\theta)+1) - \frac{\delta\rho^{(r)}(\theta)}{\delta\rho^{(r)}(\beta)}(\log\rho^{(r)}(\theta)+1)$$

$$\pm(\frac{\delta\rho(\theta)}{\delta\rho^{(r)}(\beta)} \pm \frac{\delta\rho^{(r)}(\theta)}{\delta\rho^{(r)}(\beta)})(\log(\rho(\theta)\pm\rho^{(r)}(\theta))+1)\Big)$$

$$= \int_{-\infty}^{\infty} d\theta\,\Big(\mp\frac{1}{2\pi}\varphi(\theta-\beta)(\log\rho(\theta)+1) - \delta(\theta-\beta)(\log\rho^{(r)}(\theta)+1)$$

$$\pm(\frac{1}{2\pi}\varphi(\theta-\beta) \pm \delta(\theta-\beta))(\log(\rho(\theta)\pm\rho^{(r)}(\theta))+1)\Big)$$

$$= \mp(\varphi*\log\rho)(\beta) - \log\rho^{(r)}(\beta) \pm (\varphi*\log(\rho\pm\rho^{(r)}))(\beta) + \log(\rho(\beta)\pm\rho^{(r)}(\beta))$$

$$= \pm(\varphi*\log(1\pm\frac{\rho^{(r)}}{\rho}))(\beta) + \log(\frac{\rho(\beta)}{\rho^{(r)}(\beta)}\pm 1),$$

where the upper sign (+) correspond to Bosons and the lower sign (-) corresponds to Fermions. Using the definition (2) (where (+) corresponds to Fermions and (-) to Bosons) we have that:

$$1\pm\frac{\rho^{(r)}(\beta)}{\rho(\beta)} = \frac{1}{1\pm e^{-\epsilon(\beta)}} \quad \texttt{and} \quad \frac{\rho(\beta)}{\rho^{(r)}(\beta)}\pm 1 = e^{\epsilon(\beta)}.$$

So

$$\frac{\delta s(\rho,\rho^{(r)})}{\delta\rho^{(r)}(\beta)} = \mp(\varphi*\log(1\pm e^{-\epsilon(\beta)})(\beta) + \epsilon(\beta)$$

Recalling the definition:

$$L(\theta) = \pm\log(1\pm e^{-\epsilon(\theta)})$$

for Bosons/Fermions we have that

$$\frac{\delta s(\rho,\rho^{(r)})}{\delta\rho^{(r)}(\beta)} = (\varphi*L)(\beta) + \epsilon(\beta),$$

and the minimization of the energy is then equivalent to the condition:

$$0 = \frac{\delta h(\rho^{(r)})}{\delta\rho^{(r)}(\beta)} - T\frac{\delta s(\rho,\rho^{(r)})}{\delta\rho^{(r)}(\beta)}$$

$$= m\cosh\beta - T((\varphi*L)(\beta) + \epsilon(\beta)).$$

Rearranging we obtain the TBA equation:

$$\epsilon(\beta) = \frac{m}{T}\cosh\beta - (\varphi*L)(\beta).$$

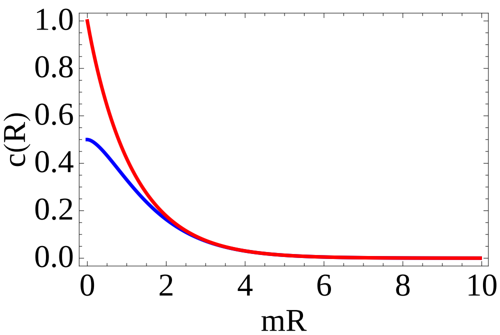
2. **Scaling functions of free theories**
   For the free Fermion and the free Boson we have that $\varphi = 0$ and $\epsilon(\theta) = mR\cosh\theta$ so the L-functions are simply:

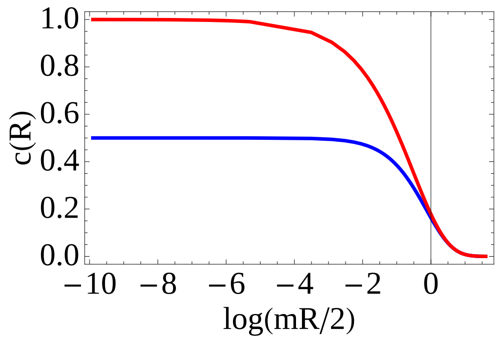
$$L(\theta) = \pm \log(1 \pm e^{-\epsilon(\theta)}) = \pm \log(1 \pm e^{-mR\cosh\theta})$$

and so the scaling function is

$$c(R) = \pm \frac{3mR}{\pi^2} \int_{-\infty}^{\infty} d\theta \, \log(1 \pm e^{-mR\cosh\theta}) \cosh\theta.$$

Note that it is a function of mR. It is of course possible to evaluate this integral numerically for different values of R. The result is (as expected):





namely, the function tends to the values 1 and 1/2 as $R \to 0$ for the free Boson/Fermion, respectively. This behaviour is easier to see when plotting the functions in a logarithmic scale. It is possible to study the $R \to 0$ analytically by expanding the logarithm in powers of $e^{-mR\cosh\theta}$. This gives the representations:

$$c(R) = \frac{3mR}{\pi^2} \sum_{k=1}^{\infty} \frac{(\mp 1)^{k+1}}{k} \int_{-\infty}^{\infty} d\theta \, e^{-mRk\cosh\theta} \cosh\theta = \frac{6mR}{\pi^2} \sum_{k=1}^{\infty} \frac{(\mp 1)^{k+1}}{k} K_1(mRk).$$

where $K_1(mRk)$ is a Bessel function. For $mR \ll 1$ we have that $K_1(mRk) \approx \frac{1}{mRk} + O(r)$ and this immediately gives the UV limit:

$$c(0) = \frac{6}{\pi^2} \sum_{k=1}^{\infty} \frac{(\mp 1)^{k+1}}{k^2} = \begin{cases} \frac{1}{2} & \texttt{for -} \\ 1 & \texttt{for +} \end{cases}$$

A detailed analysis can be found in section 6 of the paper Klassen & Melzer, NPB350 (1991) 635–689.

3. **A programme for the TBA**
   I have written a very simple Mathematica code but of course you could do this in any programming language. The code is very minimal and could be improved in many ways. For instance the number of iterations is fixed to be 50 although as you can easily observe if you make the code print for instance $L(0)$, convergence tends to happens after many fewer iterations (convergence is faster the larger R is). I am also taking $M = 250$ points (that is, values of θ) in the range $\log(mR/2)-6 < \theta < -\log(mR/2)+6$ (it can be argued that the L-functions are essentially zero outside this region). Here I have added 6 to $-\log(mR/2)$ just to make sure I am looking at a large enough interval. The programme will of course run faster if you set M to a smaller value.

```
M = 250;
Iter = 50;
m = 0;

TT = Table[{0, j}, {j, 1, 100}];

For[r = 0.001, r < 1, r = r + 0.01,
 m = m + 1;
 Rang = N[-Log[r / 2] + 6];
 the[n_] := -Rang + 2 * Rang / M * n;
 W[t_] := r Cosh[t];
 f[t_] := - (4 Sqrt[3] Cosh[t]) / (1 + 2 Cosh[2 t]);

 e = Table[W[the[n]], {n, 0, M}];
 enew = Table[W[the[n]], {n, 0, M}];

 For[j = 0, j < Iter, j++,
   L = Interpolation[Table[{the[n], Log[1 + Exp[-e[[n + 1]]]]}, {n, 0, M}]];
   For[n = 0 , n ≤ M, n++, enew[[n + 1]] = N[W[the[n]] - 1/(2 Pi) NIntegrate[f[the[n] - t] L[t], {t, -Rang, Rang}]]];
   For[n = 0 , n ≤ M, n++, e[[n + 1]] = enew[[n + 1]]];
  ]
  L = Interpolation[Table[{the[n], Log[1 + Exp[-e[[n + 1]]]]}, {n, 0, M}]];
 c = 3 r / Pi^2 NIntegrate[Cosh[t] L[t], {t, -Rang, Rang}];
 TT[[m]][[2]] = c;
 TT[[m]][[1]] = r;
]
TT
Plot[L[x], {x, -Rang, Rang}]
```

Regarding the constant TBA equation, in this case it is simply:

$$\epsilon + N \log(1 + e^{-\epsilon}) = 0,$$

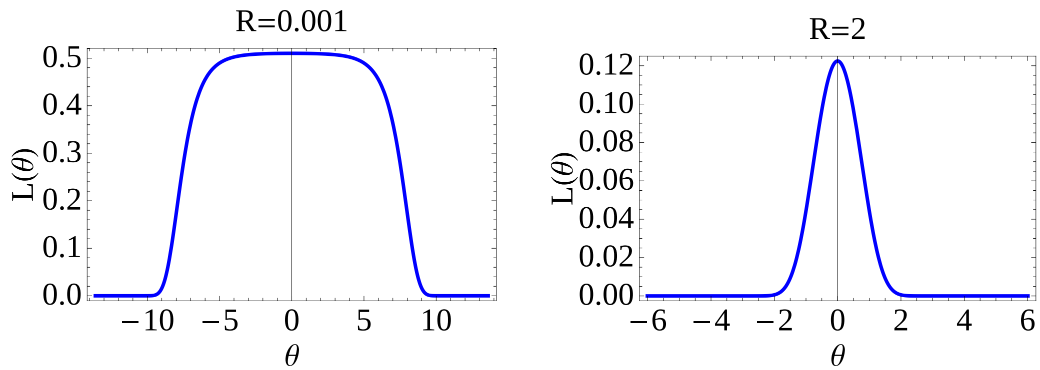
where

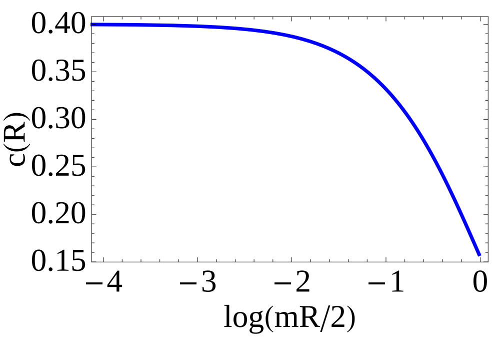
$$N = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta \, \frac{4\sqrt{3}\cosh\theta}{1 + 2\cosh 2\theta} = -1.$$

Defining $x = e^{-\epsilon}$ the equation becomes simply:

$$x = \frac{1}{1+x} \qquad \Rightarrow \qquad x = \frac{-1 \pm \sqrt{5}}{2}.$$

The only sensible solution is the one with the $+$ sign (since x is the exponential of a real quantity). This means the L-functions for R very small should develop a plateau at $L = \log(\frac{1+\sqrt{5}}{2}) \approx 0.481212$. It will look something like the first figure below. The figure on the right shows the same function for low temperatures, where the plateau disappears. The final figure shows the scaling function approaching the expected effective central charge $c_{\mathbf{eff}} = 2/5 = 0.4$. Note that in this theory this is different from the central charge which is negative.





4. **Constant TBA equations in minimal A_n Toda theory**
The constant TBA equations take the form:

$$\epsilon_a + \sum_{b=1}^{k-1} N_{ab} L_b = 0 \quad a = 1, \dots, k-1.$$

Defining $x_a = 1 + e^{-\epsilon_a}$ the equations become:

$$\prod_{b=1}^{k-1} (x_b)^{N_{ab}} = x_a - 1,$$

and substituting $N_{ab} = \delta_{ab} - 2(K^{-1})_{ab}$ we have

$$\prod_{b=1}^{n} (x_b)^{-2(K^{-1})_{ab}} = \frac{x_a - 1}{x_a}.$$

or

$$\prod_{b=1}^{n} (x_b)^{2(K^{-1})_{ab}} = \frac{x_a}{x_a - 1}.$$

In the context of the study of this kind of equations it is common to define new variables $Q_a = \prod_{b=1}^{n} (x_b)^{(K^{-1})_{ab}}$ so that $x_a = \prod_{b=1}^{n} (Q_a)^{K_{ab}}$ and the constant TBA equations then become

$$(Q_a)^2 \prod_{b=1}^{n} (Q_a)^{K_{ab}} - Q_a^2 = \prod_{b=1}^{n} (Q_a)^{K_{ab}}$$

These equations are easier to deal with because the Cartan matrix of all simply-laced algebras has a simple structure. For A_n it is $K_{ab} = 2\delta_{ab} - \delta_{a,b+1} - \delta_{a+1,b}$. This is sometimes written as $K = 2 - I$ where I is the incidence matrix with entries $I_{ab} = \delta_{a,b+1} + \delta_{a+1,b}$. So the equations for Q_a become

$$Q_a^2 Q_{a+1}^{-1} Q_{a-1}^{-1} - 1 = Q_{a+1}^{-1} Q_{a-1}^{-1}$$

and rearranging

$$Q_a^2 = 1 + Q_{a+1} Q_{a-1}. \tag{3}$$

This type of equations are simplified versions of Baxter's $T - Q$ relations which occur in the study of lattice models such as the RSOS models. Their solutions and algebraic structure has been studied in great detail specially by A. Kuniba and collaborators (see references in the web-page). One special feature of these equations is that their solutions are in fact Weyl characters associated to the corresponding Lie algebra and this holds for a wide range of integrable models whose S-matrix has an algebraic structure. We can now check whether the given solutions actually work. We had that:

$$x_a = \frac{\sin^2(\frac{\pi(a+1)}{n+3})}{\sin \frac{\pi a}{n+3} \sin \frac{\pi(a+2)}{n+3}}.$$

Since $x_a = \prod_{b=1}^n (Q_a)^{K_{ab}} = \frac{Q_a^2}{Q_{a+1}Q_{a-1}}$ we have that the solution above need satisfy:

$$\frac{\sin^2(\frac{\pi(a+1)}{n+3})}{\sin\frac{\pi a}{n+3}\sin\frac{\pi(a+2)}{n+3}} - 1 = Q_{a+1}^{-1}Q_{a-1}^{-1} = \frac{\sin^2\left(\frac{\pi}{n+3}\right)}{\sin\frac{\pi a}{n+3}\sin\frac{\pi(a+2)}{n+3}},$$

but if this holds then

$$Q_a^2 = \frac{\sin^2(\frac{\pi(a+1)}{n+3})}{\sin\frac{\pi a}{n+3}\sin\frac{\pi(a+2)}{n+3}}\left[\frac{\sin^2\left(\frac{\pi}{n+3}\right)}{\sin\frac{\pi a}{n+3}\sin\frac{\pi(a+2)}{n+3}}\right]^{-1},$$

That is $Q_a = \frac{\sin\frac{\pi(a+1)}{n+3}}{\sin\frac{\pi}{n+3}}$. Substituting this into the equation (3) we find that indeed it is satisfied.

According to the results we saw in the lecture, the central charge of the underlying CFT (in this case this is a coset) should be obtained in terms of Roger's dilogarithm function as

$$c = \frac{6}{\pi^2}\sum_{a=1}^n \mathcal{L}(1-x_a^{-1}) = \frac{6}{\pi^2}\sum_{a=1}^n \mathcal{L}(Q_a^{-2}) = \frac{6}{\pi^2}\sum_{a=1}^n \mathcal{L}\left(\frac{\sin^2\frac{\pi}{n+3}}{\sin^2\frac{\pi(a+1)}{n+3}}\right).$$

This is hard to prove analytically but can be tested numerically case-by-case. For instance for $n=1$:

$$c = \frac{6}{\pi^2}\mathcal{L}\left(\frac{1}{2}\right) = \frac{1}{2},$$

corresponding to the Ising model (A_1-minimal Toda). For $n=2$:

$$c = \frac{12}{\pi^2}\mathcal{L}\left(\frac{3-\sqrt{5}}{2}\right) = \frac{4}{5}.$$

and so on. The fact that we obtain always rational values is rather special as Roger's dilogarithm function returns complex values for most real inputs. Somehow the solutions of the constant TBA just provide the sort of inputs of Roger's dilogarithm that make the function real and proportional to π^2. This is yet another beautiful property of the solutions to the constant TBA equations!

5. **Emergence of Roger's dilogarithms from TBA equations**
Consider once more the TBA equations for a diagonal theory with a single particle. All the ideas are easy to generalise to more particles.

$$\epsilon(\theta) = mR\cosh\theta - (\varphi * L)(\theta).$$

Let us rewrite the term $mR\cosh\theta$ as:

$$mR\cosh\theta = \frac{mR}{2}(e^\theta + e^{-\theta}) = e^{\theta+\log(mR/2)} + e^{-\theta+\log(mR/2)}.$$

We are interested in the limit $mR \to 0$, that is $\log(mR/2) \to -\infty$. Suppose that we shift $\theta \to \theta - x$ with $x = \log(mR/2)$ in the TBA equation. Then it becomes

$$\epsilon(\theta - x) = e^\theta + e^{-\theta+2x} - (\varphi * L)(\theta - x),$$

for $x \to -\infty$ the term containing e^{2x} becomes negligible compared to the rest. We can also introduce "shifted" functions $\epsilon_-(\theta) := \epsilon(\theta - x)$ and $L_-(\theta) = L(\theta - x)$ and we end up with:

$$\epsilon_-(\theta) = e^\theta - (\varphi * L_-)(\theta).$$

Later on we will also need to use the derivative w.r.t. θ of this equation which gives:

$$\epsilon'_-(\theta) = e^\theta - (\varphi * L'_-)(\theta).$$

Note that the last equation is obtained after integration by parts on the convolution term. Let us now turn our attention to the scaling function

$$c(R) = \frac{3mR}{\pi^2}\int_{-\infty}^{\infty} d\theta\, L(\theta)\cosh\theta = \frac{3mR}{\pi^2}\int_0^{\infty} d\theta\,(L(\theta)+L(-\theta))\cosh\theta = \frac{6mR}{\pi^2}\int_0^{\infty} d\theta\, L(\theta)\cosh\theta,$$

where the last identity holds for parity-symmetric theories. Let us perform a similar shift:

$$\begin{aligned} c(R) &= \frac{6}{\pi^2}\int_0^{\infty} d\theta\, L(\theta)(e^{\theta+x} + e^{-\theta+x}) \\ &= \frac{6}{\pi^2}\int_x^{\infty} d\theta\, L_-(\theta)(e^{\theta} + e^{-\theta+2x}) \approx \frac{6}{\pi^2}\int_x^{\infty} d\theta\, L_-(\theta)e^\theta, \end{aligned}$$

We now replace e^θ inside this integral by its expression $e^\theta = \epsilon'_-(\theta) - (\varphi * L'_-)(\theta)$:

$$c(R) \approx \frac{3}{\pi^2}\int_x^{\infty} d\theta\, L_-(\theta)(\epsilon'_-(\theta) + (\varphi * L'_-)(\theta)).$$

The term

$$\int_x^{\infty} d\theta\, L_-(\theta)\epsilon'_-(\theta) = \int_{\epsilon_-(x)}^{\epsilon_-(\infty)} d\epsilon_-\, L_-(\theta) = \int_{\epsilon_-(x)}^{\epsilon_-(\infty)} dx\, \log(1+e^{-x}).$$

The other term can be approximated as follows:

$$\int_x^{\infty} d\theta\, L_-(\theta)(\varphi * L'_-)(\theta) \approx \int_{-\infty}^{\infty} d\theta\, L_-(\theta)(\varphi * L'_-)(\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta \int_{-\infty}^{\infty} d\beta L_-(\theta)\varphi(\theta-\beta)L'_-(\beta)$$

$$\approx \int_x^\infty d\theta L'_-(\theta)(\varphi * L_-)(\theta).$$

Here of course we used the fact that $x \to -\infty$ so we can actually replace x by $-\infty$ and viceversa. We can now use the shifted TBA equations to get rid of the convolution as $e^\theta - \epsilon_-(\theta) = (\varphi * L_-)(\theta)$ so the integral becomes:

$$\int_x^\infty d\theta\, L'_-(\theta)(\varphi * L_-)(\theta) \approx \int_x^\infty d\theta L'_-(\theta)e^\theta - \int_x^\infty d\theta L'_-(\theta)\epsilon_-(\theta).$$

We can now use the fact that

$$L'_-(\theta) = -\frac{e^{-\epsilon_-(\theta)}}{1+e^{-\epsilon_-(\theta)}}\epsilon'_-(\theta)$$

to simplify the last integral

$$-\int_x^\infty d\theta L'_-(\theta)\epsilon_-(\theta) = \int_{\epsilon_-(x)}^{\epsilon_-(\infty)} dx \frac{x}{1+e^x}.$$

Also, the integral

$$\int_x^\infty d\theta L'_-(\theta)e^\theta = -\int_x^\infty d\theta L_-(\theta)e^\theta,$$

after integration by parts and using that $\lim_{\theta\to\pm\infty} L_-(\theta)e^\theta = 0$. Going back to our original expression, we have shown that

$$c(R) = \frac{6}{\pi^2}\left(\int_{\epsilon_-(x)}^{\epsilon_-(\infty)} dx\, \log(1+e^{-x}) - \int_x^\infty d\theta L_-(\theta)e^\theta + \int_{\epsilon_-(x)}^{\epsilon_-(\infty)} dx \frac{x}{1+e^x}\right)$$

That is

$$2c(R) = \frac{6}{\pi^2}\int_{\epsilon_-(x)}^{\epsilon_-(\infty)} dx\, \left(\log(1+e^{-x}) + \frac{x}{1+e^x}\right)$$

Defining the new variable $y = (1+e^x)^{-1}$ so that $x = \log(1-y) - \log y$, $1+e^{-x} = (1-y)^{-1}$ and $dx = -\frac{dy}{y(1-y)}$ this becomes

$$2c(R) \approx \frac{6}{\pi^2}\int_{(1+e^{\epsilon_-(x)})^{-1}}^{(1+e^{\epsilon_-(\infty)})^{-1}} dx\, \left(\frac{\log(1-y)}{y} + \frac{\log y}{1-y}\right).$$

We have that $\epsilon_-(x) = \epsilon(0)$ and $\epsilon_-(\infty) = \epsilon(\infty) = \infty$, so $(1+e^{\epsilon_-(\infty)})^{-1} = 0$. This gives the final expression:

$$\lim_{R\to 0} c(R) = -\frac{3}{\pi^2}\int_0^{(1+e^{\epsilon_-(x)})^{-1}} dx\, \left(\frac{\log(1-y)}{y} + \frac{\log y}{1-y}\right) = \frac{6}{\pi^2}\mathcal{L}\left(\frac{1}{1+e^{\epsilon(0)}}\right),$$

where $\epsilon(0)$ is the solution of the constant TBA equations.