PART B

Lecture 1: Thermodynamic Bethe Ansatz Approach to Integrable Quantum Field Theory

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November 17, 2016

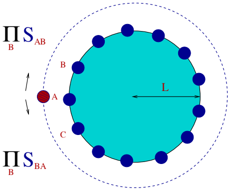
1. Motivation

- The scattering matrices of integrable quantum field theories (IQFTs) can be constructed as solutions to a set of consistency equation which almost entirely fix the S-matrix.
- However, there are certain uncertainties, such as CDD factors. So, how do we make sure that we have the correct S-matrix?
- There is strong motivation to develop consistency checks for integrable S-matrices: that is computational tools that allow us to extract information about the underlying CFT by employing just the S-matrix and particle spectrum as input.
- Both the TBA and the Form Factor Programme can be thought of consistency checks, even though they are much more than that.

- Consider an IQFT with N_A particles of species A and $A = 1, \ldots, N$
- Consider these particles to be far away from each other so that interactions are weak and the system may be described by the Bethe wave function $\Psi(x_1, \ldots, x_N)$ depending only on particle positions
- Let the system be compactified onto a circle of length L
- Then, the Bethe wave function must satisfy the following periodicity condition:
  $\Psi(x_1, \ldots, x_A = 0, \ldots, x_N) = \Psi(x_1, \ldots, x_A = L, \ldots, x_N)$
- In other words, the wave function should not change if a particle takes *a trip around the world...*

- However, when it does so, it interacts will all other particles, so periodicity of the wave function is equivalent to:

$$e^{iLM_A \sinh\theta_A} \prod_{B\neq A} S_{AB}(\theta_A - \theta_B) = 1, \quad A = 1, \ldots, N$$



- The logarithm of this equation (times $-i$) gives rise to the famous Bethe ansatz equations:

Bethe Ansatz Equations

$$LM_A \sinh\theta_A + \sum_{B\neq A} \delta_{AB}(\theta_A - \theta_B) = 2\pi n_A, \;\; \delta_{AB}(\theta) = -i\log S_{AB}(\theta)$$

3. Roots and Holes

- Much can be said about the solutions to these equations. For each particle of type A there will be a set of allowed values of $n_A = \{n_A^1, \ldots n_A^{N_A}\}$ which are integers and possibly repeating for Bosons and distinct for Fermions.
- For each set of values there will be a corresponding set of solutions to the equations $\theta_A^{(i)}$ with $A = 1, \ldots, N$ and $i = 1, \ldots, N_A$. Each such solution is called a root.
- There are also sets of values $n_A^{(i)}$ that may be skipped. The solutions $\theta_A^{(i)}$ associated to “skipped” choices of $n_A^{(i)}$ are called holes.
- In the thermodynamic or continuum limit the densities of roots and holes will play an important role.

4. Root/Hole Densities in the Thermodynamic Limit

- Let $N_A, L \to \infty$ with N_A/L finite. This is the thermodynamic limit.
- In this limit we may define root and hole densities:

$$\rho_A^{(r)}(\theta) = \frac{1}{L}\frac{dn_A^{(r)}}{d\theta} = \text{number of roots with rapidities in } [\theta, \theta+d\theta]$$

$$\rho_A^{(h)}(\theta) = \frac{1}{L}\frac{dn_A^{(h)}}{d\theta} = \text{number of holes with rapidities in } [\theta, \theta+d\theta]$$

- The total density of states per unit length is:
$\rho_A(\theta) = \rho_A^{(r)}(\theta) + \rho_A^{(h)}(\theta) = \frac{1}{L}\frac{dn_A}{d\theta}$

5. Thermodynamic Limit of the BA Equations

- In the thermodynamic limit the θ-derivative of the Bethe ansatz equations become:

$$\rho_A(\theta) = \frac{M_A}{2\pi}\cosh\theta + \sum_{B=1}^{N} \varphi_{AB} * \rho_B^{(r)}(\theta)$$

where $\varphi_{AB}(\theta) = \frac{d\delta_{AB}}{d\theta}$ and $f * g = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\beta f(\theta-\beta)g(\beta)$.

- These equations are still hard to solve but they give us a relationship between the state and root densities.
- It is convenient to define the pseudo-energies $\epsilon_A(\theta)$ as

$$\frac{\rho_A^{(r)}(\theta)}{\rho_A(\theta)} = \frac{1}{e^{\epsilon_A(\theta)} \pm 1}$$

for Fermions (+) and Bosons (-)

6. Thermodynamic Equilibrium

- The equations above can be manipulated to generate equations for the pseudo-energies $\epsilon_A(\theta)$.
- This is achieved by requiring thermodynamic equilibrium or the minimization of the free energy per unit length.
- For instance, the total energy of the system is

$$h(\rho^{(r)}) = \sum_{A=1}^{N} M_A \int_{-\infty}^{\infty} d\theta \, \rho_A^{(r)}(\theta) \cosh\theta$$

- And the total entropy:

$$s(\rho,\rho^{(r)}) = \sum_{A=1}^{N} \int_{-\infty}^{\infty} d\theta \, (\mp\rho_A \log\rho_A - \rho_A^{(r)} \log\rho_A^{(r)} \pm (\rho_A \pm \rho_A^{(r)}) \log(\rho_A \pm \rho_A^{(r)}))$$

for Bosons and Fermions

- The free energy is $f(\rho,\rho^{(r)}) = h(\rho^{(r)}) - Ts(\rho,\rho^{(r)})$. Minimization means $\frac{df}{d\rho^{(r)}} = 0$.

7. Thermodynamic Bethe Ansatz Equations

- The minimization constraint combined with the relationship between ρ_A and $\rho_A^{(r)}$ gives the TBA equations

Thermodynamic Bethe Ansatz Equations

$$\epsilon_A(\theta) = \frac{M_A}{T}\cosh\theta - \sum_{B=1}^{\infty} \varphi_{AB} * L_B(\theta), \quad A = 1, \ldots, N$$

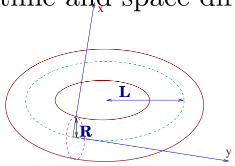
$L_A(\theta) = \pm \log(1 \pm e^{-\epsilon_A(\theta)})$ for Fermions (+) and Bosons (-).

- They are coupled integro-differential equations for $\epsilon_A(\theta)$.
- Even for complicated theories, they can be easily solved numerically through a recursive procedure.
- For free theories $\varphi_{AB}(\theta) = 0$ and $\epsilon_A(\theta) = \frac{M_A}{T}\cosh\theta$ are simply the free "on-shell" energies.
- The extremal free energy is simply

$$f(T) = -\frac{T}{2\pi}\sum_{A=1}^{N} M_A \int_{-\infty}^{\infty} d\theta \, \cosh\theta L_A(\theta)$$

8. Relation to QFT on a Torus

- Consider our QFT on a torus of circumferences L and R. We may now quantize the system in two possible ways depending on the choice of the time and space directions.



- Taking $L \to \infty$ as time direction and keeping R finite we have a compactified system whose partition function is led by the term $e^{-E_0(R)L}$ where $E_0(R)$ is the ground state energy.
- Exchanging the roles of time and space we have instead a theory where time is periodic and space infinite. If we identify $R = T^{-1}$ then we can think of this as a description of a QFT at finite temperature.
- The partition function is now dominated by the term $e^{-RLf(R)}$ where $f(R)$ is the free energy per unit length.

9. Ground State Energy and Central Charge

- It follows from the previous slide that

$$E_0(R) = Rf(R) = -\frac{1}{2\pi}\sum_{A=1}^{N} M_A \int_{-\infty}^{\infty} d\theta \, \cosh\theta L_A(\theta)$$

- The limit $R \to 0$ is the ultraviolet or high energy limit. In this limit we expect to approach the underlying CFT and so the ground state energy is related to the central charge in the usual way:

$$\lim_{R\to 0} E_0(R) = -\frac{\pi c_{\text{eff}}}{6R} \quad \text{with} \quad c_{\text{eff}} = c - 24\Delta$$

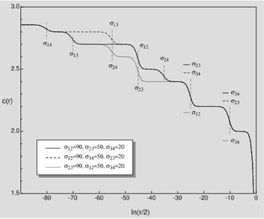
- This provides a strong consistency check of the S-matrix.
- For generic values of R we may define a scaling function:

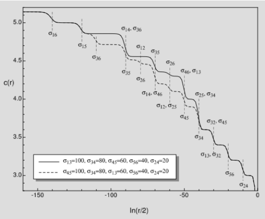
TBA Scaling Function

$$c(R) = \frac{3R}{\pi^2}\sum_{A=1}^{N} M_A \int_{-\infty}^{\infty} d\theta \, L_A(\theta)\cosh\theta.$$

10. Features of the Scaling Function

- The scaling function $c(R)$ is one of the most studied objects within the TBA approach. It admits various interpretations: as an "off-critical" Casimir energy and as a measure of the number of degrees of freedom which are excited at a particular energy scale $1/R$.
- For massive QFTs it is expected that $\lim_{R\to\infty} c(R) = 0$ and $\lim_{R\to 0} c(R) = c_{\text{eff}}$.
- The function $c(R)$ smoothly interpolates between these two values and, at least for unitary theories, is monotonically decreasing (from the UV to IR).





11. Constant TBA Equations

- Although the TBA equations can usually only be solved numerically, the UV limit of these equations can be often treated analytically and exactly.
- The equations resulting in this limit are known as constant TBA equations. They are coupled algebraic equations which often encode beautiful algebraic structures.
- Recall once more the TBA equations:

$$\epsilon_A(\theta) = M_A R \cosh\theta - \sum_{A=1}^{\infty} \varphi_{AB} * L_B(\theta), \quad A = 1, \ldots, N$$

- As $R \to 0$ the functions $\epsilon_A(\theta)$ are constant for a wide range of values of θ. The kernels $\varphi_{AB}(\theta)$ are generally strongly picked around $\theta = 0$. Let ϵ_A, L_A be these constant values.
- Then, the TBA equations become simply:

Constant TBA Equations

$$\epsilon_A + \sum_{B=1}^{N} N_{AB} L_B = 0 \quad \text{with} \quad N_{AB} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta\, \varphi_{AB}(\theta)$$

- The constant TBA equations can be exactly solved for known N_{AB}. For many theories, N_{AB} encodes algebraic information about the structure of the S-matrices.
- For instance for minimal Toda field theory: $N_{AB} = \delta_{AB} - 2K^{-1}_{AB}$ where K_{AB} is the Cartan matrix associated to a particular simply laced algebra.
- Taking the infinite temperature limit of the TBA equations and the formula for the scaling function, it is possible to show:

Central Charge and Rogers Dilogarithm

$$c_{\mathit{eff}} = \frac{6}{\pi^2} \sum_{A=1}^{N} \mathcal{L}\left(\frac{1}{1+e^{\epsilon_A}}\right)$$

where $\mathcal{L}(x) = \frac{1}{2}\int_0^x dy \left(\frac{\ln y}{y-1} - \frac{\log(1-y)}{y}\right)$ is Roger's dilogarithm function.

13. Y-Systems

- Another way of writing the TBA-equations is to express them in terms of the functions $Y_A(\theta) = e^{-\epsilon_A(\theta)}$.
- Let me illustrate this with a simple example: consider a theory with a single particle and kernel $\varphi(\theta) = 2\text{sech}\theta$.
- The TBA-equation is simply $\xi(\theta) + (\varphi * L)(\theta) = 0$ for $\xi(\theta) = \epsilon(\theta) - mR\cosh\theta$. Let us Fourier-tranform this equation:

$$\tilde{\xi}(\omega) + \tilde{\varphi}(\omega)\tilde{L}(\omega) = 0.$$

  Where the "tildes" indicate Fourier-transformed functions. In this case it is easy to show that $\tilde{\varphi}(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\theta \, \frac{e^{i\theta\omega}}{\cosh\theta} = \text{sech}\frac{\pi\omega}{2}$. So the equation becomes $\cosh\frac{\pi\omega}{2}\tilde{\xi}(\omega) + \tilde{L}(\omega) = 0$.
- Take now the inverse Fourier transform of this equation. It is possible to use the following property:

$$\int_{-\infty}^{\infty} d\omega e^{-i\theta\omega} \cosh\frac{\pi\omega}{2}\tilde{\xi}(\omega) = \frac{1}{2}\left(\xi(\theta - \frac{i\pi}{2}) + \xi(\theta + \frac{i\pi}{2})\right).$$

14. Y-Systems: Example

- So, in summary, the TBA equations have now been transformed into

$$\xi(\theta - \frac{i\pi}{2}) + \xi(\theta + \frac{i\pi}{2}) + 2L(\theta) = 0.$$

- Exponentiating both sides and using the definition $Y(\theta) = e^{-\epsilon(\theta)}$ we obtain

Sinh-Gordon Y-System

$$Y(\theta - \frac{i\pi}{2})Y(\theta + \frac{i\pi}{2}) = (1 + Y(\theta))^{-2}$$

- This is a finite-difference equation which contains the same information as the original integro-differential equation.
- Note that the explicit dependence in mR has disappeared! It is now "hidden" in the asymptotic properties of the $Y(\theta)$ functions.