Lecture 4: Twist Fields in Quantum Field Theory

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1. Twist Fields in QFT

- Twist fields are nothing new in QFT. Whenever there is an internal symmetry in QFT there generally is a symmetry field (or twist field) associated to it.
- A well-known example is the Ising model. Both as a minimal model of CFT and as a massive QFT, the Ising model has three fields 1, ϵ and μ generally known as the identiy, energy and order fields.
- The Ising model has $\mathbb{Z}_2$ symmetry and we can think of the order field μ as a twist field.
- As we saw earlier, this means that there is a non-trivial factor of local commutativity $\omega = -1$ associated with the field μ.
- Twist fields are non-local w.r.t. other fields in the theory but they are local w.r.t. the Lagrangian density of the theory (as they respect the internal symmetry).
- In this sense they are local fields and they can be studied as such.

2. Branch Point Twist Fields in QFT

- It is known since a long time that a twist field may be associated to the $\mathbb{Z}_n$ symmetry of an orbifolded CFT constructed as n cyclicly connected copies of a given CFT. The conformal dimension of such field $\mathcal{T}$ was also found in this context.

Twist Field Conformal Dimension

$$\Delta_{\mathcal{T}} = \frac{c}{24}\left(n - \frac{1}{n}\right), \quad \Delta_{:\mathcal{T}\phi:} = \frac{c_{\text{eff}}}{24}\left(n - \frac{1}{n}\right) \quad (\textbf{non}-\textbf{unitary})$$

- In the investigation of the EE a field of the "same" dimension was first identified by Calabrese and Cardy in 2004. In this work, this field was interpreted as associated to a conical singularity in the complex plane (see reference list).
- In 2008 we proposed an interpretation of this field as branch point twist field. In particular we showed that branch point twist fields are local in a replica theory.

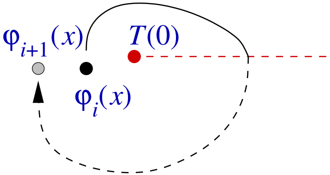
3. Exchange Relations in QFT

- Let φ be a local field of a given QFT. Consider a replica theory where n copies φ_i, $i = 1, \ldots, n$ exist and $i + n \equiv i$.
- Two Branch Point Twist Fields may be defined which are characterized by the following commutation relations:

$$\begin{aligned} \varphi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\varphi_{i+1}(y) \qquad x^1 > y^1, \\ \varphi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\varphi_i(y) \qquad x^1 < y^1, \end{aligned}$$

$$\begin{aligned} \varphi_i(y)\tilde{\mathcal{T}}(x) &= \tilde{\mathcal{T}}(x)\varphi_{i-1}(y) \qquad x^1 > y^1, \\ \varphi_i(y)\tilde{\mathcal{T}}(x) &= \tilde{\mathcal{T}}(x)\varphi_i(y) \qquad x^1 < y^1. \end{aligned}$$

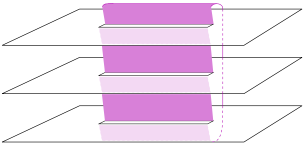
- $\mathcal{T}$ implements the cyclic permutation symmetry $i \mapsto i + 1$ and $\tilde{\mathcal{T}} = \mathcal{T}^\dagger$ implements the inverse map $i \mapsto i - 1$.



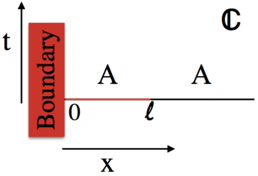
- We now have two ways of computing the EE of a single interval in CFT: we may compute the partition function Z_n on a Riemann manifold with one branch cut or we may compute a two-point function of twist fields.
- Let us start by using the first approach. Recall that $\mathrm{Tr}_{\mathcal{A}}\rho_A^n = Z_n/Z_1^n$. Points on the complex plane ω can be mapped to points z the n-sheeted Riemann manifold through the conformal map

$$z = \left(\frac{\omega}{\omega - \ell}\right)^{\frac{1}{n}}$$

The points $\omega = 0$ and $\omega = \ell$ are conical singularities.



$\Rightarrow$

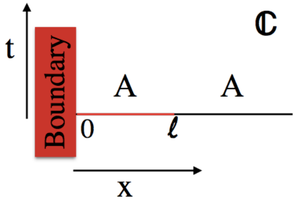


5. Computing Z_n Using Conformal Mapping

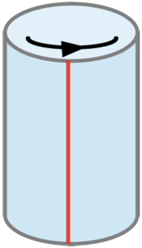
- Let us map this configuration to a cylinder. The map that achieves this is

$$\sigma = i \log\left(\frac{\ell - \omega + 2\epsilon}{\ell + \omega + 2\epsilon}\right)$$

where we introduced a cut-off ϵ so as to avoid singularities. In his set up the branch cut now runs vertically from $\sigma = 0$ to $\sigma = i \log \frac{\ell}{\epsilon}$ (time direction) and has total length $\log \frac{\ell}{\epsilon}$.



$\Rightarrow$



6. CFT on a Cylinder

- CFT on the cylinder has particularly nice properties.
- In particular, the Hamiltonian becomes simply $L_0 + \bar{L}_0 - \frac{c}{12}$ in terms of Virasoro generators and this will help us finally evaluate the partition functions.

$$Z_n = \langle e^{-\log \frac{\ell}{\epsilon} H_{\rm rep}} \rangle, \quad Z_1^n = \langle e^{-\log \frac{\ell}{\epsilon} H_0} \rangle^n$$

  where $H_{\rm rep}$ is the Hamiltonian in the replica theory and $H_0 = L_0 + \bar{L}_0 - \frac{c}{12}$ is the Hamiltonian of the original theory.
- It is easy to evaluate Z_1^n as we just need to know the lowest eigenvalue of $L_0, \bar{L}_0$. In terms of conformal maps we have the same cylinder but without the branch cut.
- In unitary theories, these eigenvalues are simply 0 but in non-unitary theories we may have a non-vanishing lowest eigenvalue $\Delta = \bar{\Delta}$ and so, in general:

$$Z_1^n \sim e^{-2n \log \frac{\ell}{\epsilon} (\Delta - \frac{c}{24})}$$

- Computing Z_n is a little harder.

7. Orbifold Theories

- The replica theory is what is what is usually called an orbifold in CFT.
- In this orbifold we can also construct a Virasoro algebra $\mathcal{L}_k$ associated with central charge c and a stress-energy tensor $T(z)=\sum_{j=1}^{n}T^{(j)}(z)$ with $T^{(j)}(z+2\pi)=T^{(j+1)}(z)$.
- The total Virasoro algebra is then a sub-algebra of $\mathcal{L}_k$ with central charge nc whose generators can be defined as:

$$L_k^{\text{rep}}=\frac{\mathcal{L}_{nk}}{n}+\Delta_{\mathcal{T}}\delta_{0,k}$$

- The Hamiltonian is then

$$H^{\text{rep}}=L_0^{\text{rep}}+\bar{L}_0^{\text{rep}}-\frac{nc}{12}.$$

- This gives

$$Z_n\sim e^{-2\log\frac{\ell}{\epsilon}\left(\frac{\Delta}{n}+\Delta_{\mathcal{T}}-\frac{nc}{24}\right)}$$

8. Finally...

- Thus we have that:

Replica Partition Function

$$\mathrm{Tr}_{\mathcal{A}}\rho_A^n = \frac{Z_n}{Z_1^n} = \left(\frac{\epsilon}{\ell}\right)^{\frac{c_{\mathrm{eff}}}{12}\left(n-\frac{1}{n}\right)} \quad \text{with} \quad c_{\mathrm{eff}} = c - 24\Delta$$

- From this expression one may easily derive the known formulae for the von Neumann and the Rényi entropies:

$$S(\ell) = \frac{c_{\mathrm{eff}}}{6}\log\frac{\ell}{\epsilon} \quad \text{and} \quad S_n(\ell) = \frac{c_{\mathrm{eff}}(n+1)}{12n}\log\frac{\ell}{\epsilon}$$

- The extra factor $1/2$ compared to previous formulae comes in because there is only one boundary point (in the calculation we assume the system starts at $x=0$).

9. EE from Branch Point Twist Fields

- The same results can be obtained much more easily by employing branch point twist fields:

Partition Function as Correlator of Twist Fields

$$\mathrm{Tr}_{\mathcal{A}}(\rho_A^n) \propto \epsilon^{4\Delta_{\mathcal{T}}} \langle \mathcal{T}(0)\tilde{\mathcal{T}}(\ell)\rangle.$$

- In CFT ($\ell \ll \xi$) such representation indeed gives the expected formulae for the EE since:

$$\epsilon^{4\Delta_{\mathcal{T}}} \langle \mathcal{T}(0)\tilde{\mathcal{T}}(\ell)\rangle = \left(\frac{\epsilon}{\ell}\right)^{4\Delta_{\mathcal{T}}} \;\Rightarrow\; S_n(\ell) \sim \frac{c(n+1)}{6n}\log\left(\frac{\ell}{\epsilon}\right)$$

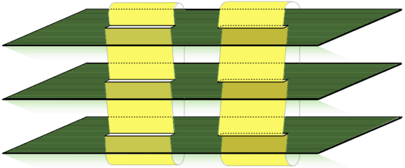
- A representation in terms of twist fields shows also saturation for large distances ($\ell \gg \xi$):

$$\lim_{\ell\to\infty} \epsilon^{4\Delta_{\mathcal{T}}} \langle \mathcal{T}(0)\tilde{\mathcal{T}}(\ell)\rangle = \epsilon^{4\Delta_{\mathcal{T}}} \langle \mathcal{T}\rangle^2 \;\Rightarrow\; S_n(\ell) \sim \frac{c(n+1)}{6n}\log\left(\frac{\epsilon}{m}\right)+U_n$$

- Saturation follows from factorization of correlators at large distances. Here $\langle \mathcal{T}\rangle = m^{2\Delta_{\mathcal{T}}} a_n$ and $U_n = \frac{\log(a_n^2)}{1-n}$.

10. Final Observations

- Note that our twist field results involve c instead of c_{eff}.
- This is because in non-unitary theories, where $c \neq c_{\text{eff}}$ another type of twist field needs to be used.
- We note that the twist field approach facilitates computations, even for the simplest case we have considered here.
- In addition, it is really the only approach that we can use for massive theories (where conformal invariance is broken) and even for CFT if the Riemann manifold is more complicated.
- For instance, for the LN:



$$\mathcal{E}[n] = \langle \mathcal{T}(0)\tilde{\mathcal{T}}(\ell_1)\tilde{\mathcal{T}}(\ell_2)\mathcal{T}(\ell_3)\rangle$$